

Random coordinate descent methods for ℓ_0 regularized convex optimization

Andrei Patrascu and Ion Necoara

Abstract

In this paper we study the minimization of ℓ_0 regularized optimization problems, where the objective function is composed of a smooth convex function and the ℓ_0 regularization. We analyze necessary optimality conditions for this nonconvex problem which lead to the separation of the local minima into two restricted classes. Based on these restricted classes of local minima, we devise new random coordinate descent type methods for solving these problems. In particular, we analyze the properties of an iterative hard thresholding based random coordinate descent algorithm for which we prove that any limit point is a local minimum from the first restricted class of local minimizers. Then, we analyze the convergence of a random proximal alternating minimization method and show that any limit point of this algorithm is a local minima from the second restricted class of local minimizers. Under the strong convexity assumption, we prove linear convergence in probability for both methods. We also provide numerical experiments which show the superior behavior of our methods in comparison with the usual iterative hard thresholding algorithm.

I. INTRODUCTION

Nowadays, there are increasingly numerous real-world applications which promote interest in sparse optimization problems. In recent practical applications (e.g. sparse control problems [16], [21], [22], state estimation under corrupted measurements [9], [14], [15], compressed sensing [4]–[7], sparse support vector machines [1], sparse nonnegative factorization [10], sparse principal component analysis [13]), we deal with a convex optimization problem for which we like to get an (approximate) solution, but we also desire a solution which has the additional property of

The authors are with Automatic Control and Systems Engineering Department, University Politehnica Bucharest, 060042 Bucharest, Romania. Corresponding author: I. Necoara, email: ion.necoara@acse.pub.ro.

sparsity (it has few nonzero components). Under certain assumptions on our original problem, we can recover a sufficiently sparse solution using appropriate algorithms. However, in general, the additional requirement of sparsity immediately turns the original convex problem into a very hard combinatorial problem, since solving it exactly would require to try all possible sparse patterns in a brute-force way.

Typically, in order to obtain a sparse minimizer of an optimization problem, the usual approach involves minimizing the number of nonzero components of the solution. Therefore, in the literature for convex sparse optimization problems two formulations are widespread:

- (i) *the regularized formulation*: $\min_{x \in \mathbb{R}^n} f(x) + \lambda \|x\|_0$, where $\|x\|_0$ is the quasinorm which counts the number of nonzero components in the vector x and $\lambda > 0$ a penalty parameter;
- (ii) *the sparsity constrained formulation*: $\min_{x \in \mathbb{R}^n} \{f(x) : \|x\|_0 \leq s\}$, where the number of nonzero elements $s \in \{1, \dots, n\}$ of the solution is fixed.

Note that both formulations are NP-hard and there is no clear equivalence between them in the general case. There have been given and repeatedly improved several greedy algorithms for the sparse least squares settings (i.e. where the smooth function to be minimized is $f(x) = \|Ax - b\|^2$) under certain restricted isometry assumptions (see e.g. [1], [4]–[7]). Amongst these algorithms for the regularized least squares settings, the iterative hard thresholding algorithm has gained a lot of interest lately due to its simple and intuitive iteration (see [4], [5]). Recently, in [17], a generalization of the iterative hard thresholding algorithm to general convex functions f has been given under the ℓ_0 regularized formulation and its iteration complexity has also been derived. In paper [17], the author analyzes the ℓ_0 regularized convex cone programming with smooth and strongly convex objective function. For general convex objective functions, the author considers the minimization over a bounded box set. However, since there could be an exponential number of local minimizers for the ℓ_0 regularized problem, there is no characterization in [17] of the local minima at which the iterative hard thresholding algorithm converges. Moreover, in the general convex case we need to consider an ℓ_0 regularized formulation with additional bounded box constraints. These issues limit the application of this algorithm to general ℓ_0 regularized optimization problems. Further, general sparsity constrained formulations were considered e.g. in [2], where the authors introduced several classes of stationary points and developed greedy coordinate descent algorithms converging to different classes of stationary points. In [19], penalty decomposition methods were devised for both regularized and constrained formulations of sparse

nonconvex problems and convergence analysis was provided for these algorithms.

On the other hand, coordinate descent methods are some of the most successful in the large-scale optimization field and used frequently to solve sparse optimization problems [2], [18], [25]. Roughly speaking, these methods are based on the strategy of updating one (block) coordinate of the vector of variables per iteration using some index selection procedure (e.g. cyclic, greedy or random). This often reduces drastically the iteration complexity and memory requirements, making these methods simple and scalable. There exist numerous papers dealing with the convergence analysis of this type of method. For example, in [27] Nesterov presented the iteration complexity of a random coordinate descent method for minimization of smooth convex functions. The complexity results from [27] have been extended to the convex composite case (and with global linear constraints) in [25], [31] ([23], [24], [26]). For nonconvex composite optimization, convergence analysis of random coordinate descent methods was given e.g. in [29]. In this paper we analyze local minima and devise coordinate descent type methods for the following ℓ_0 regularized optimization problem:

$$\min_{x \in \mathbb{R}^n} F(x) \quad (= f(x) + \|x\|_{0,\lambda}), \quad (1)$$

where function f is smooth and convex and the quasinorm of x is defined as:

$$\|x\|_{0,\lambda} = \sum_{i=1}^N \lambda_i \|x_i\|_0,$$

with $x_i \in \mathbb{R}^{n_i}$ being the i th block component of x and $\lambda_i \geq 0$ for all $i = 1, \dots, N$. Note that in this formulation we do not impose sparsity on all block components of x , but only on those i th blocks for which the corresponding penalty parameter $\lambda_i > 0$. In the first part of the paper we analyze necessary optimality conditions for problem (1) and prove that any point satisfying these conditions is a local minimum of (1). Further, we separate the set of local minima into two restricted classes around the set of global minima. In the second part of the paper, based on this separation of local minima we devise new random coordinate descent type methods for our problem (1). In particular, we introduce a random coordinate descent iterative hard thresholding (RC-IHT) method which combines a random coordinate descent scheme and the iterative hard thresholding algorithm. We analyze the convergence properties of algorithm (RC-IHT) and we prove that any limit point of the sequence generated by the algorithm is included in the first class of restricted local minima. Further, we introduce a random proximal alternating minimization

iterative hard thresholding (RPAM-IHT) method and we prove that any limit point of the sequence generated by the algorithm is included in the second class of restricted local minima. Under the strong convexity assumption, we prove linear convergence in probability for both methods. We also provide numerical experiments which show the practical improvement which our methods provide in comparison with the usual iterative hard thresholding algorithms.

Contribution. The contribution of the paper can be summarized as follows:

(i) We analyze the properties of problem (1) and derive necessary optimality conditions. We also introduce two classes of restricted local minima and show that first class is included in the second one, with both classes containing the set of global minimizers.

(ii) We combine a random coordinate gradient descent scheme and iterative hard thresholding method into a new algorithm that involves, at each iteration, the solution of a quadratic low-dimensional subproblem that is computed in closed form. We show that any limit point of this algorithm is a restricted local minimum from the first class. Furthermore, we show that the local minima reached by this new method are better, in some sense, than the local minima obtained by the classical iterative hard thresholding algorithm. We also prove linear rate of convergence in probability for this first algorithm.

(iii) We propose a variant of random proximal alternating minimization algorithm, that involves at each iteration the exact solution of a low-dimensional optimization subproblem. We show that any limit point of the second algorithm is a restricted local minimum from the second class. Furthermore, the local minima reached by this algorithm are better, in some sense, than the local minima obtained by the first algorithm. We also prove for this second algorithm linear rate of convergence in probability.

Content. The structure of the paper is as follows. In Section 2, we state the necessary optimality conditions of our problem and show that only local minima satisfy these conditions. Further we introduce two classes of restricted local minima and analyze their properties. In Section 3 we derive two random coordinate descent methods for solving problems (1). In Section 4 we analyze the global convergence properties of our methods and characterize their limit points. In Section 5 we derive iteration complexity results in probability for both methods under the strong convexity assumption. Finally, in Section 6 we provide numerical experiments which show that our algorithms perform favorably on such problems.

A. Notations and preliminaries

We consider the space \mathbb{R}^n composed by column vectors. For $x, y \in \mathbb{R}^n$ denote the scalar product by $\langle x, y \rangle = x^T y$ and the Euclidean norm by $\|x\| = \sqrt{x^T x}$. We use the same notation $\langle \cdot, \cdot \rangle$ ($\|\cdot\|$) for scalar product (norm) in spaces of different dimensions. For any matrix $A \in \mathbb{R}^{m \times n}$ we use $\text{rank}(A)$ for the rank and $\sigma_{\min}(A)$ for the minimal singular value of matrix A . We use the notation $[n] = \{1, 2, \dots, n\}$ and $e = [1 \dots 1]^T \in \mathbb{R}^n$. In the sequel, we consider the following decompositions of the variable dimension and of the $n \times n$ identity matrix:

$$n = \sum_{i=1}^N n_i, \quad I_n = [U_1 \dots U_N], \quad I_n = [U_{(1)} \dots U_{(n)}],$$

where $U_i \in \mathbb{R}^{n \times n_i}$ and $U_{(j)} \in \mathbb{R}^n$ for all $i \in [N]$ and $j \in [n]$. If the index set corresponding to block i is given by \mathcal{S}_i , then $|\mathcal{S}_i| = n_i$. Given $x \in \mathbb{R}^n$, then for any $i \in [N]$ and $j \in [n]$, we denote:

$$\begin{aligned} x_i &= U_i^T x \in \mathbb{R}^{n_i}, & \nabla_i f(x) &= U_i^T \nabla f(x) \in \mathbb{R}^{n_i}, \\ x_{(j)} &= U_{(j)}^T x \in \mathbb{R}, & \nabla_{(j)} f(x) &= U_{(j)}^T \nabla f(x) \in \mathbb{R}. \end{aligned}$$

For any vector $M \in \mathbb{R}_{++}^N$, we define the following norm and its corresponding dual:

$$\|x\|_M = \left(\sum_{i=1}^N M_i \|x_i\|^2 \right)^{1/2}, \quad \|y\|_M^* = \left(\sum_{i=1}^N \frac{1}{M_i} \|y_i\|^2 \right)^{1/2}.$$

For any vector $x \in \mathbb{R}^n$, the support of x is given by $\text{supp}(x)$, which denotes the set of indices corresponding to the nonzero components of x . We denote $\bar{x} = \max_{j \in \text{supp}(x)} |x_{(j)}|$ and $\underline{x} = \min_{j \in \text{supp}(x)} |x_{(j)}|$. Additionally, we introduce the following set of indices:

$$I(x) = \text{supp}(x) \cup \{j \in [n] : \lambda_i = 0, j \in \mathcal{S}_i\}$$

and $I^c(x) = [n] \setminus I(x)$. Given two scalars $p \geq 1, r > 0$ and $x \in \mathbb{R}^n$, the p -ball of radius r and centered in x is denoted by $\mathcal{B}_p(x, r) = \{y \in \mathbb{R}^n : \|y - x\|_p < r\}$. Let $I \subseteq [n]$ and denote the subspace of all vectors $x \in \mathbb{R}^n$ satisfying $I(x) \subseteq I$ with S_I , i.e. $S_I = \{x \in \mathbb{R}^n : x_i = 0 \ \forall i \notin I\}$. We denote with f^* the optimal value of the convex function f , i.e. $f^* = \min_{x \in \mathbb{R}^n} f(x)$ and the optimal set of function f with $X_f^* = \{x \in \mathbb{R}^n : \nabla f(x) = 0\}$.

In this paper we consider the following assumption on optimization problem (1):

Assumption 1.1: The function f has (block) coordinatewise Lipschitz continuous gradient with constants $L_i > 0$ for all $i \in [N]$, i.e. the convex function f satisfies the following inequality:

$$\|\nabla_i f(x + U_i h_i) - \nabla_i f(x)\| \leq L_i \|h_i\| \quad \forall x \in \mathbb{R}^n, h_i \in \mathbb{R}^{n_i}.$$

An immediate consequence of Assumption 1.1 is the following relation [27]:

$$f(x + U_i h_i) \leq f(x) + \langle \nabla_i f(x), h_i \rangle + \frac{L_i}{2} \|h_i\|^2 \quad \forall x \in \mathbb{R}^n, h_i \in \mathbb{R}^{n_i}. \quad (2)$$

We denote with $L = [L_1 \cdots L_N]^T \in \mathbb{R}^N$ and with L_f the global Lipschitz constant of the gradient $\nabla f(x)$. In the Euclidean settings, under Assumption 1.1 a tight upper bound of the global Lipschitz constant is $L_f \leq \sum_i L_i$ (see [27, Lemma 2]). Note that a global inequality based on L_f , similar to (2), can be also derived. Also, note that the Assumption 1.1 has been frequently considered in coordinate descent settings (see e.g. [23]–[27], [31]).

II. NECESSARY OPTIMALITY CONDITIONS

In this section we present the necessary optimality conditions for problem (1) and provide a detailed description of local minimizers. First, we establish conditions on λ for obtaining trivial global minimizers and necessary optimality conditions satisfied by any local minimum. Then, we separate the set of local minima into two restricted classes and we show that first class is included in the second one, with both classes containing the set of global minimizers.

The next theorem represents a nontrivial generalization of the results for the least squares settings presented in [30], i.e. when $f(x) = \|Ax - b\|^2$.

Theorem 2.1: Under Assumption 1.1 the following relations hold:

- (i) For any $y, d \in \mathbb{R}^n$ satisfying $y \neq 0$ and $d \in \mathcal{B}_\infty(0, \rho) \setminus S_{I(y)}$, with the radius $\rho = \min \left\{ \underline{y}, \frac{\lambda}{\|\nabla f(y)\|_1} \right\}$, the following inequality is valid: $F(y + d) \geq F(y)$.
- (ii) Any $z \in \mathbb{R}^n \setminus \{0\}$ is a local minimizer of problem (1) on $\mathcal{B}_\infty(z, \rho)$, with $\rho = \min \left\{ \underline{z}, \frac{\lambda}{\|\nabla f(z)\|_1} \right\}$, if and only if z is a global minimizer of convex problem $\min_{x \in S_{I(z)}} f(x)$.
- (iii) Let $z_0 = \arg \min_{x: \|x\|_{0,\lambda}=0} f(x)$. If the vector $\lambda \in \mathbb{R}^N$ satisfies $\underline{\lambda} \geq f(z_0) - f^*$, then the vector z_0 is a global minimizer of problem (1). Otherwise, it is at least a local minimizer on $\mathcal{B}_\infty(z_0, \rho)$, with $\rho = \frac{\lambda}{\|\nabla f(z_0)\|_1}$.
- (iv) Let $\lambda \in \mathbb{R}_+^N$ be the form $\lambda = \tilde{\lambda} \tilde{e}$, with $\tilde{e} \in \mathbb{R}^N$ having entries 0 – 1 and $\text{supp}(\tilde{e}) = \text{supp}(\lambda)$. Then, $z = \arg \min_{x \in X_f^*} \|x\|_{0,\tilde{e}}$ satisfying $\|z\|_{0,\tilde{e}} > 0$ is a global minimizer of problem (1) provided that we have $\tilde{\lambda} \leq \min_{\|x\|_{0,\tilde{e}} \leq \|z\|_{0,\tilde{e}} - 1} \frac{f(x) - f^*}{\|z\|_{0,\tilde{e}} - \|x\|_{0,\tilde{e}}}$.

Proof: (i) Let $y, d \in \mathbb{R}^n$, with $y \neq 0$ and assume $\|d\|_\infty < \underline{y}$. Then, we have:

$$|y_{(i)} + d_{(i)}| \geq |y_{(i)}| - |d_{(i)}| \geq \underline{y} - \|d\|_\infty > 0 \quad \forall i \in \text{supp}(y). \quad (3)$$

Clearly, for any $d \in \mathcal{B}_\infty(0, \rho) \setminus S_{I(y)}$, with $\rho = \underline{y}$, we have:

$$\|y + d\|_{0,\lambda} = \|y\|_{0,\lambda} + \sum_{i \in I^c(y) \cap \text{supp}(d)} \|d_{(i)}\|_{0,\lambda} \geq \|y\|_{0,\lambda} + \underline{\lambda}.$$

Let $d \in \mathcal{B}_\infty(0, \rho) \setminus S_{I(y)}$, with $\rho = \min \left\{ \underline{y}, \frac{\underline{\lambda}}{\|\nabla f(y)\|_1} \right\}$. The convexity of function f and the Holder inequality lead to:

$$\begin{aligned} F(y + d) &\geq f(y) + \langle \nabla f(y), d \rangle + \|y + d\|_{0,\lambda} \\ &\geq F(y) - \|\nabla f(y)\|_1 \|d\|_\infty + \underline{\lambda} \geq F(y) \quad \forall y \in \mathbb{R}^n. \end{aligned}$$

(ii) For the first implication, we assume that z is a local minimizer of function F on the open ball $\mathcal{B}_\infty(z, \rho)$, i.e. we have:

$$f(z) \leq f(y) \quad \forall y \in \mathcal{B}_\infty(z, \rho) \cap S_{I(z)}.$$

Since f has also global Lipschitz continuous gradient, with constant L_f , we derive (see (2)):

$$f(z) \leq f(y) \leq f(z) + \langle \nabla f(z), y - z \rangle + \frac{L_f}{2} \|y - z\|^2 \quad \forall y \in \mathcal{B}_\infty(z, \rho) \cap S_{I(z)}.$$

Taking $\alpha = \min \left\{ \frac{1}{L_f}, \frac{\rho}{\max_{j \in I(z)} |\nabla_{(j)} f(z)|} \right\}$ and $y = z - \alpha \nabla_{I(z)} f(z)$, we have:

$$0 \leq \left(\frac{\alpha^2}{2L_f} - \frac{\alpha}{L_f} \right) \|\nabla_{I(z)} f(z)\|^2 \leq 0.$$

Therefore, we have $\nabla_{I(z)} f(z) = 0$, which means that:

$$z = \arg \min_{x \in S_{I(z)}} f(x). \quad (4)$$

For the second implication, we assume that z satisfies (4). For any $x \in \mathcal{B}_\infty(z, \rho) \cap S_{I(z)}$ we have $\|x - z\|_\infty < \underline{z}$, which by (3) implies that $|x_{(i)}| > 0$ whenever $|z_{(i)}| > 0$. Therefore, we get:

$$F(x) = f(x) + \|x\|_{0,\lambda} \geq f(z) + \|z\|_{0,\lambda} = F(z),$$

and combining with the result in (i) we have proved the implication.

(iii) For any $x \in \mathbb{R}^n$ satisfying $\|x\|_{0,\lambda} = 0$, it can be easily seen that

$$F(x) = f(x) \geq f(z_0) = F(z_0).$$

On the other hand, assume $\|x\|_{0,\lambda} > 0$ and $\underline{\lambda} \geq f(z_0) - f^*$. Then, we have:

$$F(x) = f(x) + \|x\|_{0,\lambda} \geq f(x) - f^* + f(z_0) \geq F(z_0).$$

In conclusion, under assumption $\underline{\lambda} \geq f(z_0) - f^*$, we have $F(x) \geq F(z_0)$ for all x . In the case when the relation $\underline{\lambda} < f(z_0) - f^*$ holds, then from the convexity of f we get:

$$F(x) \geq f(z_0) + \langle \nabla f(z_0), x \rangle + \|x\|_{0,\lambda} \quad \forall x \in \mathbb{R}^n.$$

Applying the Holder inequality we have:

$$F(x) \geq F(z_0) - \|\nabla f(z_0)\|_1 \|x\|_\infty + \underline{\lambda} \quad \forall x \in \mathbb{R}^n.$$

In conclusion, for $x \in \mathcal{B}_\infty\left(z_0, \frac{\underline{\lambda}}{\|\nabla f(z_0)\|_1}\right)$, we have $F(x) \geq F(z_0)$, i.e. z_0 is a local minimum.

(iv) Let $\|z\|_{0,\bar{e}} > 0$ and note that for any $y \in \mathbb{R}^n$ satisfying $\|y\|_{0,\bar{e}} \geq \|z\|_{0,\bar{e}}$ we have:

$$F(y) = f(y) + \|y\|_{0,\lambda} \geq f^* + \|z\|_{0,\lambda} = F(z).$$

On the other hand, let $\|y\|_{0,\bar{e}} < \|z\|_{0,\bar{e}}$ and assume $\tilde{\lambda} \leq \min_{\|x\|_{0,\bar{e}} \leq \|z\|_{0,\bar{e}}-1} \frac{f(x)-f^*}{\|z^*\|_{0,\bar{e}}-\|x\|_{0,\bar{e}}}$, which implies:

$$\tilde{\lambda} \leq \min_{\|x\|_{0,\bar{e}} \leq \|z\|_{0,\bar{e}}-1} \frac{f(x) - f^*}{\|z\|_{0,\bar{e}} - \|x\|_{0,\bar{e}}} \leq \frac{f(y) - f^*}{\|z\|_{0,\bar{e}} - \|y\|_{0,\bar{e}}}. \quad (5)$$

We observe that:

$$\begin{aligned} F(y) &= f(y) + \|y\|_{0,\lambda} \geq f(y) + \|y\|_{0,\lambda} - \|z\|_{0,\lambda} + \|z\|_{0,\lambda} \\ &= f(y) + \tilde{\lambda} (\|y\|_{0,\bar{e}} - \|z\|_{0,\bar{e}}) + \|z\|_{0,\lambda} \\ &\stackrel{(5)}{\geq} f(y) + (f^* - f(y)) + \|z\|_{0,\lambda} = F(z), \end{aligned}$$

and thus we conclude that $F(y) \geq F(z)$ for all $y \in \mathbb{R}^n$. ■

Independently of parameters λ_i , Theorem 2.1 (ii) provides conditions for obtaining local minimizers of function F (see the corollary below).

Corollary 2.2: Under Assumption 1.1, any vector $z \in \mathbb{R}^n$ is a local minimizer of problem (1) if and only if the following equality holds:

$$\nabla_{I(z)} f(z) = 0.$$

Proof: Solving a problem instance $\min_{x \in S_I} f(x)$ is equivalent with finding a vector $z \in S_I$ with $I(z) = I$ and $\nabla_{I(z)} f(z) = 0$. Then, from Theorem 2.1 (ii) we get the result. ■

It is not hard to see that when the function f is strongly convex, the number of local minima of problem (1) is finite, otherwise we might have an infinite number of local minimizers.

A. Strong local minimizers

In this section we introduce the notion of basic local minimizers and then we define two restricted classes of local minimizers corresponding to problem (1).

Definition 2.3: A vector z is called a *basic local minimizer* for problem (1) if it satisfies:

$$\nabla_{I(z)} f(z) = 0.$$

We denote with \mathcal{T}_f the class of all basic local minimizers of function f . It can be easily seen that finding a basic local minimizer is a trivial procedure e.g.: (i) if we choose some set of indices $I \subseteq [n]$ such that $\{j \in [n] : j \in \mathcal{S}_i, \lambda_i = 0\} \subseteq I$, then from Theorem 2.1 (ii) the minimizer of the convex problem $\min_{x \in S_I} f(x)$ is a basic local minimizer for problem (1), i.e. it satisfies the condition from Definition 2.3; (ii) if we minimize the convex function f w.r.t. all blocks i satisfying $\lambda_i = 0$, then from Theorem 2.1 (iii) we obtain again some basic local minimizer for (1). This motivates us to introduce more restricted classes of local minimizers.

Definition 2.4: Given $M \in \mathbb{R}_{++}^N$, a vector z is called an *M-strong local minimizer* for problem (1) if it satisfies the following two conditions:

- (i) z is a basic local minimizer, i.e. $\nabla_{I(z)} f(z) = 0$
- (ii) and additionally satisfies:
$$\begin{cases} |\nabla_{(j)} f(z)| \leq \sqrt{2\lambda_i M_i}, & \text{if } z_{(j)} = 0 \\ |z_{(j)}| \geq \sqrt{\frac{2\lambda_i}{M_i}}, & \text{if } z_{(j)} \neq 0, \quad \forall i \in [N] \text{ and } j \in \mathcal{S}_i. \end{cases}$$

We denote with $\mathcal{M}(M)$ the class of all M -strong local minimizers of problem (1). Note that the vector M determines how restrictive the corresponding class of M -strong local minimizers is. Given two vectors $M^1, M^2 \in \mathbb{R}_{++}^N$, with $M_i^1 \leq M_i^2$ for all $i \in [N]$, it can be easily seen from Definition 2.4 (ii) that $\mathcal{M}(M^1) \subseteq \mathcal{M}(M^2)$.

We now introduce a second class of strong local minimizers.

Definition 2.5: Given $\beta \in \mathbb{R}_{++}^N$, a vector z is called a β -coordinatewise local minimizer for problem (1) if it satisfies the following conditions:

$$F(z) \leq \min_{h_i \in \mathbb{R}^{n_i}} F(z + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2 \quad \forall i \in [N].$$

We denote with $\mathcal{L}(\beta)$ the class of all β -coordinatewise local minimizers of function F . We also denote the set of global minimizers for problem (1) with \mathcal{X}^* . We further describe the relationships between the previous two classes of restricted local minima.

Theorem 2.6: Let Assumption 1.1 hold and $\beta \in \mathbb{R}_{++}^N$, then the following inclusions are valid:

$$\mathcal{X}^* \subseteq \mathcal{L}(\beta) \subseteq \mathcal{M}(L + \beta) \subseteq \mathcal{T}_f.$$

Proof: (i) Assume $z \in \mathcal{X}^*$, i.e. it is a global minimizer of function F . Then, we have:

$$F(z) \leq \min_{h_i \in \mathbb{R}^{n_i}} F(z + U_i h_i) \leq \min_{h_i \in \mathbb{R}^{n_i}} F(z + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2 \quad \forall \beta_i > 0, i \in [N],$$

and thus $z \in \mathcal{L}(\beta)$. We further take $z \in \mathcal{L}(\beta)$ and using the Lipschitz property of the gradient of f given in (2) we get:

$$\begin{aligned} F(z) &\leq F(z + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2 \\ &\leq F(z) + \langle \nabla_i f(z), h_i \rangle + \frac{L_i + \beta_i}{2} \|h_i\|^2 + \lambda_i (\|z_i + h_i\|_0 - \|z_i\|_0) \end{aligned}$$

for any $h_i \in \mathbb{R}^{n_i}$ and $i \in [N]$. First, note that choosing $h_i = -\frac{1}{L_i + \beta_i} U_{(j)} \nabla_{(j)} f(z)$ for $j \in I(z) \cap \mathcal{S}_i$, we have $\nabla_{(j)} f(z) = 0$. Therefore, z satisfies $\nabla_{I(z)} f(z) = 0$. Secondly, if we take $h_i = -U_{(j)} z_{(j)}$ for any $j \in I(z) \cap \mathcal{S}_i$, we obtain $|z_{(j)}|^2 \geq 2\lambda_i / (L_i + \beta_i)$. On the other hand, if we take $h_i = -\frac{1}{L_i + \beta_i} U_{(j)} \nabla_{(j)} f(z)$ for any $j \in I^c(z) \cap \mathcal{S}_i$, we obtain $|\nabla_{(j)} f(z)|^2 \leq 2\lambda_i (L_i + \beta_i)$, which proves the second inclusion. The third inclusion is straightforward from Definition 2.4 (i). ■

From Theorem 2.6 we observe that the M -strong local minimizers for problem (1) are included in the class of all basic local minimizers. Thus, designing an algorithm which converges to a local minimum from $\mathcal{M}(M)$ will be of interest. Similarly, the β -coordinatewise minimizers form a more restricted class of local minima than $\mathcal{M}(L + \beta)$ and therefore, designing an algorithm which converges to a local minimum from $\mathcal{L}(\beta)$ will be of more interest. To illustrate the relationships between the previously defined classes of restricted local minima and see how much they are related to global minima, let us consider an example.

Example 2.7 Consider $f(x) = \frac{1}{2} \|Ax - b\|^2$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ satisfying:

$$A = J_{m,3} \begin{bmatrix} 1 \cdots n \\ 2 \cdots n + 1 \\ 3 \cdots n + 2 \end{bmatrix} + [pI_m \quad O_{m,n-m}], \quad b = qJ_{m,1},$$

with $J_{m,n} \in \mathbb{R}^{m \times n}$ the matrix having all entries 1. We choose the following parameter values: $m = 4, n = 5, p = 5, q = 65, \lambda = 2$ and $\beta_i = 0.1$ for all $i \in [n]$ and the results are given in Table I. From 32 possible local minima, we found 11 local minimizers in $\mathcal{M}(L_{fe})$ and only 4

local minimizers in $\mathcal{M}(L)$. Moreover, the class of β -coordinatewise minima $\mathcal{L}(\beta)$ contains only one vector which is also the global optimum of problem (1), i.e. in this case $\mathcal{L}(\beta) = \mathcal{X}^*$. From Table I we can clearly see that the newly introduced classes of local minimizers are much more restricted (in the sense of having small number of elements, close to that of the set of global minimizers) than the class of basic local minimizers that is much larger.

TABLE I
LOCAL MINIMA DISTRIBUTION

| Class of local minima | \mathcal{T}_f | $\mathcal{M}(L_f e)$ | $\mathcal{M}(L)$ | $\mathcal{L}(\beta)$ |
|------------------------|-----------------|----------------------|------------------|----------------------|
| Number of local minima | 32 | 11 | 4 | 1 |

Remark 2.8 Assume the least squares settings, i.e. $f(x) = \|Ax - b\|^2$, with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $\lambda = \tilde{\lambda}e$, with $\tilde{\lambda} > 0$. For any $z \in \mathcal{T}_f$, from Definition 2.3 we have:

$$A_{I(z)}^T (Az - b) = 0.$$

Note that if $\|z\|_0 \geq \text{rank}(A)$, then $z \in X_f^*$ and $f(z) = f^*$. Therefore, assuming that a global minimizer x^* of $F(x)$ satisfies $x^* \notin X_f^*$, then we get $\|x^*\|_0 \leq \text{rank}(A)$. On the other hand, assume that the global minimizer x^* belongs to X_f^* , then it can be easily seen that $\|x^*\|_0 = \min_{x \in X_f^*} \|x\|_0 \leq \text{rank}(A)$. In conclusion, we obtained that any global minimizer in the least squares settings satisfies $\|x^*\|_0 \leq \text{rank}(A)$. \square

Motivated by Remark 2.8, we now describe the vectors from the two new classes of restricted minima, $\mathcal{M}(L)$ and $\mathcal{L}(\beta)$, respectively, by the largest gap in terms of function values measured w.r.t. the “best” local minimum with the same sparsity. For simplicity, we consider $\lambda_i = \lambda$ for those $\lambda_i > 0$, with $i \in [N]$.

Theorem 2.9: Let Assumption 1.1 hold. Additionally, we assume that f is in the form $f(x) = g(Ax) + \langle b, x \rangle$, where $g(t)$ is a σ -strongly convex function and $A \in \mathbb{R}^{m \times n}$ is any matrix with $m < n$. Let $M, \beta \in \mathbb{R}_{++}^N$, $\zeta = \max_i \left(\beta_i + \frac{M_i}{\sigma \|A_i\|^2} \right)$ and $f^*(x) = \{\min_y f(y) : \|y\|_0 = \|x\|_0\}$. Then, if z is a local minimum with $\|z\|_0 < \text{rank}(A)$ and $J = I(z)$ we get:

(i) For $z \in \mathcal{M}(M)$ and $r = \max_{|I|=|J|} \frac{\sigma}{\sigma_{\min}(A_I)} \|A_I^T A_{J \setminus I} z_{J \setminus I}\|^2$ we have:

$$f(z) - f^*(z) \leq \min\{\|z\|_0, n - \|z\|_0\} \lambda \left(\max_{|I|=|J|} \frac{\bar{M}}{\sigma \sigma_{\min}(A_I)} - \frac{\sigma \sigma_{\min}(A_{J \setminus I})}{\bar{M}} \right) + r.$$

(ii) For $z \in \mathcal{L}(\beta)$ and $r = \max_{|I|=|J|} \frac{\sigma}{\sigma_{\min}(A_I)} \|A_I^T A_{J \setminus I} z_{J \setminus I}\|^2$ we have:

$$f(z) - f^*(z) \leq \min\{\|z\|_0, n - \|z\|_0\} \lambda \left(\max_{|I|=|J|} \frac{\bar{M}}{\sigma \sigma_{\min}(A_I)} - \frac{\sigma \sigma_{\min}(A_{J \setminus I})}{\zeta} \right) + r.$$

Proof: (i) Using the strong convexity property of function g and Definition 2.4, we have:

$$F(x) \geq F(z) + \langle \nabla f(z), x \rangle + \frac{\sigma}{2} \|A(x - z)\|^2 + (\|x\|_{0,\lambda} - \|z\|_{0,\lambda}) \quad x \in \mathbb{R}^n.$$

Minimizing both sides w.r.t. the subspace $S = \{x \in \mathbb{R}^n : \|x\|_0 = \|z\|_0\}$, we obtain:

$$\min_{x \in S} F(x) \geq \min_{x \in S} F(z) + \langle \nabla f(z), x \rangle + \frac{\sigma}{2} \|A(x - z)\|^2.$$

We denote the support of vector x with I , the support of z with J , $\mathcal{J} = J \setminus I$, $\mathcal{I} = I \setminus J$ and $\mathcal{C} = J \cap I$. Based on the strong convexity property of function g we have:

$$\begin{aligned} f^*(z) &\geq \min_{x \in S} f(z) + \langle \nabla_{\mathcal{I}} f(z), x_{\mathcal{I}} \rangle + \frac{\sigma}{2} \|A_I(x - z) + A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &= \min_{x \in S} f(z) + \langle \nabla_{\mathcal{I}} f(z), x_{\mathcal{I}} \rangle + \frac{\sigma}{2} \|A_I(x - z)\|^2 + \frac{\sigma}{2} \|A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &\quad + \sigma \langle x_I - z_I, A_I^T A_{\mathcal{J}} z_{\mathcal{J}} \rangle \\ &= \min_{x \in S} f(z) + \langle \nabla_{\mathcal{I}} f(z) + \sigma A_{\mathcal{I}}^T A_{\mathcal{J}} z_{\mathcal{J}}, x_{\mathcal{I}} \rangle + \frac{\sigma}{2} \|A_I(x - z)\|^2 + \frac{\sigma}{2} \|A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &\quad + \sigma \langle x_{\mathcal{C}} - z_{\mathcal{C}}, A_{\mathcal{C}}^T A_{\mathcal{J}} z_{\mathcal{J}} \rangle \\ &\geq \min_{x \in S} f(z) + \langle \nabla_{\mathcal{I}} f(z) + \sigma A_{\mathcal{I}}^T A_{\mathcal{J}} z_{\mathcal{J}}, x_{\mathcal{I}} \rangle + \frac{\sigma \sigma_{\min}(A_I)}{2} \|x_{\mathcal{I}}\|^2 + \frac{\sigma}{2} \|A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &\quad + \sigma \langle x_{\mathcal{C}} - z_{\mathcal{C}}, A_{\mathcal{C}}^T A_{\mathcal{J}} z_{\mathcal{J}} \rangle + \frac{\sigma \sigma_{\min}(A_I)}{2} \|x_{\mathcal{C}} - z_{\mathcal{C}}\|^2 \\ &\geq \min_{I \subseteq [n], |I|=\|z\|_0} f(z) - \frac{1}{2\sigma \sigma_{\min}(A_I)} \|\nabla_{\mathcal{I}} f(z) + \sigma A_{\mathcal{I}}^T A_{\mathcal{J}} z_{\mathcal{J}}\|^2 + \frac{\sigma \sigma_{\min}(A_{\mathcal{J}})}{2} \|z_{\mathcal{J}}\|^2 \\ &\quad - \frac{\sigma}{2\sigma_{\min}(A_I)} \|A_{\mathcal{C}}^T A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &\geq \min_{I \subseteq [n], |I|=\|z\|_0} f(z) - \frac{1}{\sigma \sigma_{\min}(A_I)} \|\nabla_{\mathcal{I}} f(z)\|^2 - \frac{\sigma}{\sigma_{\min}(A_I)} \|A_{\mathcal{I}}^T A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &\quad + \frac{\sigma \sigma_{\min}(A_{\mathcal{J}})}{2} \|z_{\mathcal{J}}\|^2 - \frac{\sigma}{2\sigma_{\min}(A_I)} \|A_{\mathcal{C}}^T A_{\mathcal{J}} z_{\mathcal{J}}\|^2 \\ &\geq \min_{I \subseteq [n], |I|=\|z\|_0} f(z) - \frac{1}{\sigma \sigma_{\min}(A_I)} \sum_{j \in \mathcal{I} \cap \mathcal{S}_i} 2\lambda_i M_i + \sigma \sigma_{\min}(A_{\mathcal{J}}) \sum_{j \in \mathcal{J} \cap \mathcal{S}_i} \frac{\lambda_i}{M_i} \\ &\quad - \frac{\sigma}{\sigma_{\min}(A_I)} \|A_{\mathcal{I}}^T A_{\mathcal{J}} z_{\mathcal{J}}\|^2, \end{aligned} \tag{6}$$

where in the last inequality we used Definition 2.4. Note that for any subset satisfying $|I| = |J|$ we have $|I \setminus J| = |J \setminus I| = |\mathcal{I}|$. Since $|\mathcal{I}| \leq \min\{n - |J|, |J|\}$, we can derive the above result.

(ii) Let $z \in \mathcal{L}(\beta)$, $i \in [N]$ and $j \in I(z) \cap \mathcal{S}_i$. Then, we have:

$$F(z) \leq \min_{h_i \in \mathbb{R}^{n_i}} F(z + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2 \leq F(z - U_{(j)} z_{(j)}) + \frac{\beta_i}{2} \|z_{(j)}\|^2.$$

Subtracting $\|z - U_{(j)} z_{(j)}\|_{0,\lambda}$ from both sides we get:

$$f(z) + \lambda_i \leq f(z - U_{(j)} z_{(j)}) + \frac{\beta_i}{2} \|z_{(j)}\|^2. \quad (7)$$

From the strong convexity property of function g we have:

$$\begin{aligned} f(z) &= \min_{y \in \mathbb{R}} f(z + U_{(j)}(y - z_{(j)})) \\ &\geq \min_{y \in \mathbb{R}} f(z - U_{(j)} z_{(j)}) + \langle \nabla_{(j)} f(z - U_{(j)} z_{(j)}), y \rangle + \frac{\sigma}{2} \|A_{(j)} y_{(j)}\|^2 \\ &\geq f(z - U_{(j)} z_{(j)}) - \frac{1}{2\sigma \|A_{(j)}\|^2} \|\nabla_{(j)} f(z - U_{(j)} z_{(j)})\|^2 \\ &\geq f(z - U_{(j)} z_{(j)}) - \frac{L_i}{2\sigma \|A_{(j)}\|^2} \|z_{(j)}\|^2, \end{aligned} \quad (8)$$

where in the last inequality we used the Lipschitz gradient relation $\|\nabla_{(j)} f(z - U_{(j)} z_{(j)})\| = \|\nabla_{(j)} f(z - U_{(j)} z_{(j)}) - \nabla_{(j)} f(z)\| \leq L_i \|z_{(j)}\|$ for all $j \in I(z) \cap \mathcal{S}_i$. From (7) and (8) we obtain the following bound:

$$\|z_{(j)}\|^2 \geq \frac{2\lambda_i}{\beta_i + \frac{L_i}{\sigma \|A_{(j)}\|^2}}.$$

From previous bound and the inequality (6) we obtain the result. ■

There are recent results in [25] where the convergence rate of coordinate descent methods is analyzed under the assumptions stated in Theorem 2.9. Note that we can get a better interpretation of Theorem 2.9, in terms of the condition number of the problem, when f is strongly convex.

Corollary 2.10: Under the assumptions of Theorem 2.9, with $A = I_n$, we have:

(i) If $z \in \mathcal{M}(M)$, then

$$f(z) - f^*(z) \leq \lambda \min\{n - \|z\|_0, \|z\|_0\} \left(\frac{\bar{M}}{\sigma} - \frac{\sigma}{\bar{M}} \right).$$

(ii) If $z \in \mathcal{L}(\beta)$, then

$$f(z) - f^*(z) \leq \lambda \min\{n - \|z\|_0, \|z\|_0\} \left(\frac{\bar{M}}{\sigma} - \frac{\sigma}{\bar{\beta} + \frac{\bar{M}}{\sigma \min_i \|A_i\|^2}} \right).$$

Proof: The results are implied by the fact that I_n is an orthogonal matrix, which ensures that $r = 0$ and $\sigma_{\min}(A_I) = 1$ for all $I \subseteq [n]$. ■

From Corollary 2.10 we observe that for a relatively well-conditioned function f , given a local minimizer $z \in \mathcal{M}(L)$ or $z \in \mathcal{L}(\beta)$, the support of z is one of the “best” coordinates combination of all possible subsets $\{I \subset [n] : |I| = \|z\|_0\}$ in terms of the function values.

III. RANDOM COORDINATE DESCENT TYPE METHODS

In this section we present two random coordinate descent methods suitable for solving the class of problems (1). The first algorithm is a combination of an iterative hard thresholding scheme and a random coordinate descent method, which consists in solving small dimensional ℓ_0 regularized quadratic problems at each iteration that have closed form solution. The second algorithm is also a coordinate descent method obtained from applying a random alternating minimization method to the objective function F and an additional proximal term.

Let $M \in \mathbb{R}_{++}^N$, $x \in \mathbb{R}^n$ and $i \in [N]$. Then, we introduce the following *thresholding map*:

$$T_i^M(x) = \arg \min_{y_i \in \mathbb{R}^{n_i}} f(x) + \langle \nabla_i f(x), y_i - x_i \rangle + \frac{M_i}{2} \|y_i - x_i\|^2 + \lambda_i \|y_i\|_0.$$

In order to find a local minimizer of problem (1), we introduce the *random coordinate descent iterative hard thresholding* (RC-IHT) method, whose iteration is described as follows:

ALGORITHM **(RC-IHT)**.

1. Choose $M_i \geq L_i \ \forall i \in [N]$. For $k \geq 0$ do:
2. Choose a (block) coordinate $i_k \in [N]$ with uniform probability
3. Set $x_{i_k}^{k+1} = T_{i_k}^M(x^k)$ and $x_i^{k+1} = x_i^k \ \forall i \neq i_k$.

Given $\nabla_{i_k} f(x^k)$, we can easily compute the closed form solution of $T_{i_k}^M(x^k)$ and thus the iteration of (RC-IHT) method becomes:

$$x_{(j)}^{k+1} = \begin{cases} x_{(j)}^k - \frac{1}{M_{i_k}} \nabla_{(j)} f(x^k), & \text{if } |x_{(j)}^k - \frac{1}{M_{i_k}} \nabla_{(j)} f(x^k)| \geq \sqrt{\frac{2\lambda_{i_k}}{M_{i_k}}} \\ 0, & \text{if } |x_{(j)}^k - \frac{1}{M_{i_k}} \nabla_{(j)} f(x^k)| \leq \sqrt{\frac{2\lambda_{i_k}}{M_{i_k}}}, \end{cases} \quad \forall j \in \mathcal{S}_{i_k}.$$

Note that if at some iteration $\lambda_{i_k} = 0$, then the iteration of algorithm (RC-IHT) is identical with the iteration of the usual random coordinate descent method [25], [27]. Further, our algorithm has similarities with the iterative hard thresholding algorithm (IHTA) in [17]. For completeness, we also present the algorithm (IHTA).

ALGORITHM **(IHTA)**.

1. Choose $M_f > L_f$. For $k \geq 0$ do:
2. $x^{k+1} = \arg \min_{y \in \mathbb{R}^n} f(x^k) + \langle \nabla f(x^k), y - x^k \rangle + \frac{M_f}{2} \|y - x^k\|^2 + \|y\|_{0,\lambda}$.

Note that the arithmetic complexity of computing the next iterate x^{k+1} in (RC-IHT), once $\nabla_{i_k} f(x^k)$ is known, is of order $\mathcal{O}(n_{i_k})$, which, for $N \gg 1$, is much lower than the arithmetic complexity $\mathcal{O}(n)$ of (IHTA), that additionally requires the computation of full gradient $\nabla f(x^k)$.

Remark 3.1 For $i \in [N]$ and $j \in \mathcal{S}_i$ we can see that any fixed point z_i of the thresholding map $T_i^M(x)$, i.e. any $z_i = T_i^M(z)$ satisfies:

- (i) $z_{(j)} \nabla_{(j)} f(z) = 0$;
- (ii) $\begin{cases} |\nabla_{(j)} f(z)| \leq \sqrt{2M_i \lambda_i}, & \text{if } z_{(j)} = 0, \\ |z_{(j)}| \geq \sqrt{\frac{2\lambda_i}{M_i}}, & \text{if } z_{(j)} \neq 0. \end{cases}$

Comparing these relations with the definition of M -strong local minimizers (see Definition 2.4), we conclude that any vector z , whose block components z_i are fixed points of the map $T_i^M(x)$ for all $i \in [N]$, belongs to the restricted class of local minimizers $\mathcal{M}(M)$.

Inspired by recent complexity results for alternating minimization methods, we introduce a second algorithm named *randomized proximal alternating minimization iterative hard thresholding* (RPAM-IHT) for solving problem (1). For a vector $\beta \in \mathbb{R}_{++}^N$ and $i \in [N]$ we introduce the map:

$$V_i^\beta(x) = \arg \min_{y_i \in \mathbb{R}^{n_i}} f(x + U_i(y_i - x_i)) + \frac{\beta_i}{2} \|y_i - x_i\|^2 + \lambda_i \|y_i\|_0.$$

The iteration of (RPAM-IHT) is described as follows:

ALGORITHM **(RPAM-IHT)**.

1. Choose $\beta_i > 0 \forall i \in [N]$. For $k \geq 0$ do:
2. Choose a (block) coordinate $i_k \in [N]$ with uniform probability
3. Set $x_{i_k}^{k+1} = V_{i_k}^\beta(x^k)$ and $x_i^{k+1} = x_i^k$ for all $i \neq i_k$.

Note that at each iteration of our second algorithm we need to perform an exact minimization of the objective function f w.r.t. one randomly chosen (block) coordinate. If $\lambda_{i_k} = 0$, then the iteration of algorithm (RPAM-IHT) requires solving a small dimensional subproblem with a strongly convex objective function as in the classical proximal alternating minimization algorithm. In the case when $\lambda_{i_k} > 0$ and $n_{i_k} \gg 1$, this subproblem is nonconvex and usually

hard to solve. However, for certain particular cases of the function f and $n_i = 1$ (i.e. scalar case), we can easily compute the solution of the subproblem in algorithm (RPAM-IHT). Indeed, for $x \in \mathbb{R}^n$ let us define:

$$v^i(x) = x + U_i h_i^*, \text{ where } h_i^* = \arg \min_{h_i \in \mathbb{R}} f(x + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2$$

$$\Delta^i(x) = f(x - U_i x_i) + \frac{\beta_i}{2} \|x_i\|^2 - f(v^i(x)) - \frac{\beta_i}{2} \|v^i(x)_i - x_i\|^2.$$

Then, it can be seen that the iteration of (RPAM-IHT) in the scalar case has the following form:

$$x_{i_k}^{k+1} = \begin{cases} (v^{i_k}(x^k))_{i_k}, & \text{if } \Delta^{i_k}(x^k) \geq \lambda_{i_k} \\ 0, & \text{if } \Delta^{i_k}(x^k) \leq \lambda_{i_k}. \end{cases}$$

In general, if the function f satisfies Assumption 1.1, computing $v^{i_k}(x^k)$ at each iteration of (RPAM-IHT) requires the minimization of an unidimensional, convex, and smooth function, which can be efficiently performed using unidimensional search algorithms. Further, we analyze the least squares settings to highlight the simplicity of the iteration of algorithm (RPAM-IHT) in the scalar case.

Example 3.2 Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $f(x) = \frac{1}{2} \|Ax - b\|^2$. In this case

$$\Delta^i(x) = \frac{1}{2} \|r - A_i x_i\|^2 + \frac{\beta_i}{2} \|x_i\|^2 - \frac{1}{2} \left\| r \left(I_m - \frac{A_i A_i^T}{\|A_i\|^2 + \beta_i} \right) \right\|^2 - \frac{\beta_i}{2} \left\| \frac{A_i^T r}{\|A_i\|^2 + \beta_i} \right\|^2,$$

where $r = Ax - b$. Under these circumstances, the iteration of (RPAM-IHT) will have the following form:

$$x_{i_k}^{k+1} = \begin{cases} x_{i_k}^k - \frac{A_{i_k}^T r^k}{\|A_{i_k}\|^2 + \beta_{i_k}}, & \text{if } \Delta^{i_k}(x^k) \geq \lambda_{i_k} \\ 0, & \text{if } \Delta^{i_k}(x^k) \leq \lambda_{i_k}. \end{cases} \quad (9)$$

Remark 3.3 Note that any fixed point z of the map $V_i^\beta(x)$, i.e. with $z_i = V_i^\beta(z)$, satisfies:

$$F(z) \leq \min_{h_i \in \mathbb{R}^{n_i}} F(z + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2 \quad i \in [N].$$

Comparing this relation with the definition of β -coordinatewise local minimizers (see Definition 2.5), we conclude that any vector z , whose block components z_i are fixed points of the map $V_i^\beta(x)$ for all $i \in [N]$, belongs to the restricted class of local minimizers $\mathcal{L}(\beta)$.

In the sequel we use the following notations for the entire history of index choices, expected value of objective function f w.r.t. the entire history and for the support of sequence x^k :

$$\xi^k = \{i_0, \dots, i_{k-1}\}, \quad f^k = \mathbb{E}[f(x^k)], \quad I^k = I(x^k).$$

Due to the randomness of algorithms (RC-IHT) and (RPAM-IHT), at any iteration k with $\lambda_{i_k} > 0$, the sequence I^k changes if one of the following situations holds for some $j \in \mathcal{S}_{i_k}$:

$$\begin{array}{ll} (i) \ x_{(j)}^k = 0 \text{ and } (T_{i_k}^M(x^k))_{(j)} \neq 0 & (i) \ x_{(j)}^k = 0 \text{ and } (V_{i_k}^\beta(x^k))_{(j)} \neq 0 \\ (ii) \ x_{(j)}^k \neq 0 \text{ and } (T_{i_k}^M(x^k))_{(j)} = 0 & \text{or} \quad (ii) \ x_{(j)}^k \neq 0 \text{ and } (V_{i_k}^\beta(x^k))_{(j)} = 0. \end{array}$$

In other terms, at a given moment k with $\lambda_{i_k} > 0$, we expect no change in the sequence I^k of algorithm (RC-IHT) or (RPAM-IHT) if there is no index $j \in \mathcal{S}_{i_k}$ satisfying the above corresponding set of relations (i) and (ii). We define the common notion of *change of I^k in expectation* at iteration k , for both algorithms (RC-IHT) and (RPAM-IHT). Let x^k be the sequence generated by (RC-IHT) and (RPAM-IHT), then the sequence $I^k = I(x^k)$ changes in expectation if the following situation occurs:

$$\mathbb{E}[|I^{k+1}/I^k| + |I^k/I^{k+1}| | x^k] > 0, \quad (10)$$

which implies:

$$\mathbb{P}(|I^{k+1}/I^k| + |I^k/I^{k+1}| > 0) \geq \frac{1}{N}.$$

In the next section we show that there is a finite number of changes of I^k in expectation generated by algorithms (RC-IHT) and (RPAM-IHT) and then, we prove global convergence of these two new algorithms.

IV. GLOBAL CONVERGENCE ANALYSIS

In this section we analyze the descent properties of the previously introduced coordinate descent algorithms under Assumption 1.1. Based on these properties, we establish the nature of the limit points of sequences generated by Algorithms (RC-IHT) and (RPAM-IHT). For the sequence generated by the first algorithm (RC-IHT), we derive that any accumulation point is almost surely a local minimum which belongs to $\mathcal{M}(M)$, where $M > L$. Secondly, any accumulation point of the sequence generated by algorithm (RPAM-IHT) is almost surely a local minimizer from $\mathcal{L}(\beta)$, where $\beta > 0$. Note that the classical results for any iterative algorithm used for solving

nonconvex problems state global convergence to stationary points, while for our algorithms we prove global convergence to local minima of our nonconvex and NP-hard problem (1).

In order to prove almost sure convergence results for our algorithms, we use the following supermartingale convergence lemma of Robbins and Siegmund (see e.g. [29]):

Lemma 4.1: Let v_k, u_k and α_k be three sequences of nonnegative random variables such that

$$\mathbb{E}[v_{k+1}|\mathcal{F}_k] \leq (1 + \alpha_k)v_k - u_k \quad \forall k \geq 0 \quad \text{a.s.} \quad \text{and} \quad \sum_{k=0}^{\infty} \alpha_k < \infty \quad \text{a.s.},$$

where \mathcal{F}_k denotes the collections $v_0, \dots, v_k, u_0, \dots, u_k, \alpha_0, \dots, \alpha_k$. Then, we have $\lim_{k \rightarrow \infty} v_k = v$ for a random variable $v \geq 0$ a.s. and $\sum_{k=0}^{\infty} u_k < \infty$ a.s.

A. Global convergence for algorithm (RC-IHT)

First, we derive some descent inequalities for algorithm (RC-IHT).

Lemma 4.2: Let x^k be the sequence generated by algorithm (RC-IHT). Under Assumption 1.1 the following statements hold:

(i) For $M_i \geq L_i$ we have the following descent relation:

$$\mathbb{E}[F(x^{k+1})|x^k] \leq F(x^k) - \frac{1}{2N} (\|\nabla_{I^k} f(x^k)\|_M^*)^2.$$

(ii) For $M_i > L_i$ we have the following descent relation:

$$\mathbb{E}[F(x^{k+1})|x^k] \leq F(x^k) - \mathbb{E} \left[\frac{M_{i_k} - L_{i_k}}{2} \|x^{k+1} - x^k\|^2 | x^k \right].$$

Proof: For simplicity of the exposition, we drop counter k and we use the notations:

$$x^+ = x^{k+1}, x = x^k, i = i_k \quad \text{and} \quad I = I^k.$$

(i) Let $M_i \geq L_i$, then from Assumption 1.1 and the definition of iteration x^+ of algorithm (RC-IHT) we have:

$$\begin{aligned} F(x^+) &\leq \min_{y_i \in \mathbb{R}^{n_i}} F(x) + \langle \nabla_i f(x), y_i - x_i \rangle + \frac{M_i}{2} \|y_i - x_i\|^2 + \lambda_i (\|y_i\|_0 - \|x_i\|_0) \\ &= F(x) + \sum_{j \in \mathcal{S}_i} \min \left\{ \lambda_i, \frac{M_i}{2} \|x_{(j)} - (1/M_i) \nabla_{(j)} f(x)\|^2 \right\} - (1/2M_i) \|\nabla_{(j)} f(x)\|^2 - \lambda_i \|x_{(j)}\|_0 \\ &\leq F(x) - \frac{1}{2M_i} \sum_{j \in \mathcal{S}_i \cap I} \|\nabla_{(j)} f(x)\|^2. \end{aligned}$$

Taking expectation w.r.t. i we obtain part (i) of our lemma.

(ii) Let $M_i > L_i$, then from Assumption 1.1 we have:

$$\begin{aligned}
F(x^+) &\leq \min_{y_i \in \mathbb{R}^{n_i}} F(x) + \langle \nabla_i f(x), y_i - x_i \rangle + \frac{L_i}{2} \|y_i - x_i\|^2 + \lambda_i (\|y_i\|_0 - \|x_i\|_0) \\
&= \min_{y_i \in \mathbb{R}^{n_i}} F(x) + \langle \nabla_i f(x), y_i - x_i \rangle + \frac{M_i}{2} \|y_i - x_i\|^2 + \lambda_i (\|y_i\|_0 - \|x_i\|_0) - \frac{M_i - L_i}{2} \|y_i - x_i\|^2 \\
&\leq F(x) + \langle \nabla_i f(x), x_i^+ - x_i \rangle + \frac{M_i}{2} \|x_i^+ - x_i\|^2 + \lambda_i (\|x_i^+\|_0 - \|x_i\|_0) - \frac{M_i - L_i}{2} \|x_i^+ - x_i\|^2 \\
&= \min_{y_i \in \mathbb{R}^{n_i}} \left(F(x) + \langle \nabla_i f(x), y_i - x_i \rangle + \frac{M_i}{2} \|y_i - x_i\|^2 + \lambda_i (\|y_i\|_0 - \|x_i\|_0) \right) - \frac{M_i - L_i}{2} \|x_i^+ - x_i\|^2 \\
&\leq F(x) - \frac{M_i - L_i}{2} \|x_i^+ - x_i\|^2,
\end{aligned} \tag{11}$$

where the last inequality holds by considering $y_i = x_i$. Taking expectation w.r.t. i we obtain part (ii) of our lemma. \blacksquare

We now state one of the main results of this section, namely the global convergence of algorithm (RC-IHT), by proving that there are a finite number of changes of sequence I^k in expectation.

Theorem 4.3: Let x^k be the sequence generated by algorithm (RC-IHT), with $M > L$. Then, under Assumption 1.1 the following statements hold:

(i) The sequence x^k satisfies: $\sum_{k=0}^{\infty} \|\nabla_{I^k} f(x^k)\| < \infty$ a.s.

(ii) There exists \tilde{F} such that:

$$\lim_{k \rightarrow \infty} F(x^k) = \tilde{F} \text{ a.s.} \quad \text{and} \quad \lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0 \text{ a.s.}$$

(iii) At any change in expectation of I^k we have:

$$\mathbb{E} \left[\frac{M_{i_k} - L_{i_k}}{2} \|x^{k+1} - x^k\|^2 | x^k \right] \geq \delta > 0,$$

$$\text{where } \delta = \frac{1}{N} \min \left\{ \min_{i \in [N]: \lambda_i > 0} \lambda_i \left(1 - \frac{L_i}{M_i} \right), \min_{j \in I^0} (M_i - L_i) \frac{|x_{(j)}^0|^2}{2} \right\}.$$

(iv) The sequence I^k changes in expectation a finite number of times almost surely as $k \rightarrow \infty$.

The sequence $\|x^k\|_0$ converges to some $\|x^*\|_0$ almost surely. Furthermore, any limit point of the sequence x^k belongs to $\mathcal{M}(M)$ almost surely.

Proof: (i) Using Lemma 4.1 and Lemma 4.2 (i), we can easily derive this result.

(ii) Combining Lemma 4.1 and Lemma 4.2(ii), we can ensure that

$$\lim_{k \rightarrow \infty} F(x^k) - F^* = \theta \text{ a.s.}$$

for a random variable $\theta \geq 0$ and thus $\tilde{F} = \theta + F^*$. Further, due to almost sure convergence of sequence $F(x^k)$, it can be easily seen that $\lim_{k \rightarrow \infty} F(x^k) - F(x^{k+1}) = 0$ a.s. Then, the descent

inequality (Lemma 4.2) and the assumption that $M_i > L_i$ for all $i \in [N]$ imply:

$$\frac{M_{i_k} - L_{i_k}}{2} \|x^{k+1} - x^k\|^2 \leq F(x^{k+1}) - F(x^k),$$

which leads us to $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$ a.s.

(iii) First observe that, given $x \in \mathbb{R}^n$ and $i \in [N]$, from definition of $T_i^M(x)$, we have $|(T_i^M(x))_{(j)}|^2 \geq 2\lambda_i/M_i$ whenever $(T_i^M(x))_{(j)} \neq 0$ and $j \in \mathcal{S}_i$. Replacing $x = x^k$ for $k \geq 0$, it can be easily seen that, for any $j \in \mathcal{S}_i$ and $i \in [N]$, we have

$$|x_{(j)}^k|^2 \begin{cases} \geq \frac{2\lambda_i}{M_i}, & \text{if } x_{(j)}^k \neq 0 \text{ and } i \in \xi^k \\ = |x_{(j)}^0|^2, & \text{if } x_{(j)}^k \neq 0 \text{ and } i \notin \xi^k. \end{cases}$$

Further, assume at iteration $k > 0$ that some change of sequence I^k in expectation occurs. From the definition of change of I^k in expectation, we obtain that there exists an index $j \in [n]$ (and block i containing the index j) such that either $(x_{(j)}^k = 0 \text{ and } (T_i^M(x^k))_{(j)} \neq 0)$ or $(x_{(j)}^k \neq 0 \text{ and } (T_i^M(x^k))_{(j)} = 0)$. Analyzing these cases, we have:

$$\|T_i^M(x^k) - x_i^k\|^2 \geq |(T_i^M(x^k))_{(j)} - x_{(j)}^k|^2 \begin{cases} \geq \frac{2\lambda_i}{M_i}, & \text{if } x_{(j)}^k = 0 \\ \geq \frac{2\lambda_i}{M_i}, & \text{if } x_{(j)}^k \neq 0 \text{ and } i \in \xi^k \\ = |x_{(j)}^0|^2, & \text{if } x_{(j)}^k \neq 0 \text{ and } i \notin \xi^k. \end{cases}$$

Observing that under uniform probabilities we have:

$$\mathbb{E} \left[\frac{M_{i_k} - L_{i_k}}{2} \|x^{k+1} - x^k\|^2 \mid x^k \right] = \frac{1}{N} \sum_{i=1}^N \frac{M_i - L_i}{2} \|T_i^M(x^k) - x_i^k\|^2,$$

we can conclude that at each change of I^k in expectation we get:

$$\mathbb{E} \left[\frac{M_{i_k} - L_{i_k}}{2} \|x^{k+1} - x^k\|^2 \mid x^k \right] \geq \frac{1}{N} \min \left\{ \min_{i \in [N]: \lambda_i > 0} \lambda_i \left(1 - \frac{L_i}{M_i} \right), \min_{j \in I^0} (M_i - L_i) \frac{|x_{(j)}^0|^2}{2} \right\}.$$

(iv) From $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$ a.s. we have $\lim_{k \rightarrow \infty} \mathbb{E} [\|x^{k+1} - x^k\| \mid x^k] = 0$ a.s. On the other hand from part (iii) we have that if the sequence I^k changes in expectation, then $\mathbb{E}[\|x^{k+1} - x^k\|^2 \mid x^k] \geq \delta > 0$. These facts imply that there are a finite number of changes in expectation of sequence I^k , i.e. there exist $K > 0$ such that for any $k > K$ we have $I^k = I^{k+1}$.

Further, if the sequence I^k is constant for $k > K$, then we have $I^k = I^*$ and $\|x^k\|_{0,\lambda} = \|x^*\|_{0,\lambda}$ for any vector x^* satisfying $I(x^*) = I^*$. Further, for $k > K$ algorithm (RC-IHT) is equivalent

with the usual Nesterov's random coordinate gradient descent method [27], and thus shares its convergence properties, in particular any limit point of the sequence x^k is a minimizer on the coordinates I^* for $\min_{x \in S_{I^*}} f(x)$.

Finally, assume that x^* is a limit point and for simplicity we consider that the entire sequence x^k converges to x^* . Using the definition of the thresholding map $T_i^M(x)$ and taking into account that $\|x^{k+1} - x^k\| \rightarrow 0$ a.s., we have:

$$\begin{aligned} & \mathbb{E} \left[f(x^k) + \langle \nabla f(x^k), x^{k+1} - x^k \rangle + \frac{1}{2} \|x^{k+1} - x^k\|_M^2 + \|x^{k+1}\|_{0,\lambda} \mid x^k \right], \\ & \leq f(x^k) + \frac{1}{N} \left(\langle \nabla f(x^k), x - x^k \rangle + \frac{1}{2} \|x - x^k\|_M^2 + \|x\|_{0,\lambda} + (N-1) \|x^k\|_{0,\lambda} \right), \end{aligned}$$

for all $x \in \mathbb{R}^n$. Taking the limit as $k \rightarrow \infty$ and using that $\|x^k\|_{0,\lambda} = \|x^*\|_{0,\lambda}$ as $k \rightarrow \infty$, results in:

$$F(x^*) \leq f(x^*) + \frac{1}{N} \left(\langle \nabla f(x^*), x - x^* \rangle + \frac{1}{2} \|x - x^*\|_M^2 + \|x\|_{0,\lambda} + (N-1) \|x^*\|_{0,\lambda} \right),$$

for all $x \in \mathbb{R}^n$ almost surely and thus x^* is the minimizer of the previous right hand side expression. Since this right hand side expression is block separable, we have that for any $i \in [n]$:

$$T_i^M(x^*) = \arg \min_{x_i \in \mathbb{R}^{n_i}} \langle \nabla_i f(x^*), x_i - x_i^* \rangle + \frac{M_i}{2} \|x_i - x_i^*\|^2 + \lambda_i \|x_i\|_0 = x_i^*,$$

and using the definition of local minimizers from the set $\mathcal{M}(M)$ we can conclude our statement from the theorem. ■

We have proved that any limit point x^* of our algorithm (RC-IHT) belongs to $\mathcal{M}(M)$, i.e. x^* satisfies for all $j \in \mathcal{S}_i$ and $i \in [N]$:

$$\begin{aligned} (i) \quad & x_{(j)}^* \nabla_{(j)} f(x^*) = 0; \\ (ii) \quad & \begin{cases} |\nabla_{(j)} f(x^*)| \leq \sqrt{2M_i \lambda_i}, & \text{if } x_{(j)}^* = 0, \\ |x_{(j)}^*| \geq \sqrt{\frac{2\lambda_i}{M_i}}, & \text{if } x_{(j)}^* \neq 0. \end{cases} \end{aligned}$$

It is not hard to see from [17, Theorem 3.4] that, given $M_f > L_f$, any limit point y^* of the sequence generated by algorithm (IHTA) belongs to $y^* \in \mathcal{M}(M_f e)$, i.e. y^* satisfies for all $j \in [n]$:

$$\begin{aligned} (i) \quad & y_{(j)}^* \nabla_{(j)} f(y^*) = 0; \\ (ii) \quad & \begin{cases} |\nabla_{(j)} f(y^*)| \leq \sqrt{2M_f \lambda_i}, & \text{if } y_{(j)}^* = 0, \\ |y_{(j)}^*| \geq \sqrt{\frac{2\lambda_i}{M_f}}, & \text{if } y_{(j)}^* \neq 0. \end{cases} \end{aligned}$$

Taking $M_i \approx L_i$ and $M_f \approx L_f$, we have that $M_i \leq M_f$ for all $i \in [N]$, which implies that the set of limit points of (RC-IHT) is a subset of the limit points of (IHTA), i.e. our first algorithm identifies a more restricted class of local minimizers than (IHTA) (see also Example 2.7).

B. Global convergence for algorithm (RPAM-IHT)

Further, we analyze the convergence properties of algorithm (RPAM-IHT). First, we derive a descent inequality for this algorithm.

Lemma 4.4: Let x^k be the sequence generated by (RPAM-IHT) algorithm with parameters $\beta_i > 0$ for all $i \in [N]$. Under Assumption 1.1 the following descent inequality holds:

$$\mathbb{E}[F(x^{k+1}) \mid x^k] \leq F(x^k) - \frac{1}{2N} (\|\nabla_{I^k} f(x^k)\|_{(L+\beta)}^*)^2 - \mathbb{E} \left[\frac{\beta_{i_k}}{2} \|x^{k+1} - x^k\|^2 \mid x^k \right].$$

Proof: For simplicity of exposition, we drop counter k and we use notations $x^+ = x^{k+1}$, $x = x^k$, $i = i_k$. From Assumption 1.1 we have:

$$\begin{aligned} F(x^+) + \frac{\beta_i}{2} \|x_i^+ - x_i\|^2 &= \min_{y_i \in \mathbb{R}^{n_i}} F(x + U_i(y_i - x_i)) + \frac{\beta_i}{2} \|y_i - x_i\|^2 \\ &\leq \min_{y_i \in \mathbb{R}^{n_i}} F(x) + \langle \nabla_i f(x), y_i - x_i \rangle + \frac{L_i + \beta_i}{2} \|y_i - x_i\|^2 + \lambda_i (\|y_i\|_0 - \|x_i\|_0) \\ &= F(x) + \sum_{j \in \mathcal{S}_i} \min \left\{ \lambda_i, \frac{L_i + \beta_i}{2} \|x_{(j)} - \frac{1}{L_i + \beta_i} \nabla_{(j)} f(x)\|^2 \right\} - \frac{1}{2(L_i + \beta_i)} \|\nabla_{(j)} f(x)\|^2 - \lambda_i \|x_{(j)}\|_0 \\ &\leq F(x) - \frac{1}{2(L_i + \beta_i)} \sum_{j \in \mathcal{S}_i \cap I^k} \|\nabla_{(j)} f(x)\|^2. \end{aligned}$$

Taking expectation w.r.t. i we get our descent inequality. ■

We now prove the global convergence of the sequence generated by algorithm (RPAM-IHT) to local minima which belongs to the restricted set of local minimizers $\mathcal{L}(\beta)$.

Theorem 4.5: Let x^k be the sequence generated by algorithm (RPAM-IHT) with parameters $\beta_i > 0$ for all $i \in [N]$. Under Assumption 1.1 the following statements hold:

- (i) The sequence x^k satisfies $\sum_{k=0}^{\infty} \|\nabla_{I^k} f(x^k)\| < \infty$ a.s.
- (ii) There exists \tilde{F} such that $\lim_{k \rightarrow \infty} F(x^{k+1}) - \tilde{F} = 0$ a.s. and $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$ a.s.
- (iii) At each change of sequence I^k in expectation we have the following relation:

$$\mathbb{E} [\|x^{k+1} - x^k\|^2 \mid x^k] \geq \delta > 0,$$

- where $\delta = \frac{1}{N} \min \left\{ \min_{i \in [N]: \lambda_i > 0} \lambda_i \left(1 - \frac{L_i}{L_i + \beta_i} \right), \min_{j \in I^0} \beta_j \frac{|x_{(j)}^0|^2}{2} \right\}$.
- (iv) The sequence I^k changes a finite number of times as $k \rightarrow \infty$ almost surely. The sequence $\|x^k\|_0$ converges to some $\|x^*\|_0$ almost surely. Furthermore, any limit point of the sequence x^k belongs to $\mathcal{L}(\beta)$ almost surely.

Proof: The proof for (i) and (ii) is very similar with the one of Theorem 4.3.

(iii) First, we show that any nonzero component of the sequence x^k is bounded below by a positive constant. Let $x \in \mathbb{R}^n, \beta \in \mathbb{R}_{++}^N$ and $i \in [N]$. From definition of $V_i^\beta(x)$, note that for any $j \in \text{supp}(V_i^\beta(x))$, the component $(V_i^\beta(x))_{(j)}$ is the minimizer w.r.t. j th coordinate of the function $f(x + U_i(y_i - x_i)) + \frac{\beta_i}{2} \|y_i - x_i\|^2$. Therefore, denoting $x^+ = x + U_i(V_i^\beta(x) - x_i)$, for any $j \in \text{supp}(V_i^\beta(x))$ the following optimality condition holds:

$$\nabla_{(j)} f(x^+) + \beta_i (x_{(j)}^+ - x_{(j)}) = 0.$$

On the other hand, given $j \in \text{supp}(V_i^\beta(x))$, from the definition of $V_i^\beta(x)$ we get:

$$F(x^+) + \frac{\beta_i}{2} \|x_i^+ - x_i\|^2 \leq F(x^+ - U_{(j)} x_{(j)}^+) + \frac{\beta_i}{2} \|x_i^+ - x_i - U_{(j)} x_{(j)}^+\|^2.$$

Subtracting $\|x^+ - U_{(j)} x_{(j)}^+\|_{0,\lambda} + \frac{\beta_i}{2} \|x_i^+ - x_i\|^2$ from both sides, leads to:

$$f(x^+) + \lambda_i \leq f(x^+ - U_{(j)} x_{(j)}^+) - \frac{\beta_i}{2} \|x_{(j)}^+ - x_{(j)}\|^2 + \frac{\beta_i}{2} \|x_{(j)}\|^2. \quad (12)$$

Further, if we apply the Lipschitz gradient relation in the right hand side and use the optimality conditions for the unconstrained problem solved at each iteration, we get:

$$\begin{aligned} & f(x^+ - U_{(j)} x_{(j)}^+) - \frac{\beta_i}{2} \|x_{(j)}^+ - x_{(j)}\|^2 + \frac{\beta_i}{2} \|x_{(j)}\|^2 \\ & \leq f(x^+) - \langle \nabla_{(j)} f(x^+), x_{(j)}^+ \rangle + \frac{L_i}{2} \|x_{(j)}^+\|^2 + \frac{\beta_i}{2} \|x_{(j)}\|^2 - \frac{\beta_i}{2} \|x_{(j)}^+ - x_{(j)}\|^2 \\ & = f(x^+) - \langle \beta_i (x_{(j)} - x_{(j)}^+), x_{(j)}^+ \rangle + \frac{L_i}{2} \|x_{(j)}^+\|^2 + \frac{\beta_i}{2} \|x_{(j)}\|^2 - \frac{\beta_i}{2} \|x_{(j)}^+ - x_{(j)}\|^2 \\ & = f(x^+) + \beta_i \|x_{(j)}^+\|^2 - \beta_i \langle x_{(j)}, x_{(j)}^+ \rangle + \frac{L_i}{2} \|x_{(j)}^+\|^2 + \frac{\beta_i}{2} \|x_{(j)}\|^2 - \frac{\beta_i}{2} \|x_{(j)}^+ - x_{(j)}\|^2 \\ & = f(x^+) + \frac{L_i + \beta_i}{2} \|x_{(j)}^+\|^2. \end{aligned}$$

Combining with the left hand side of (12), we get

$$\|(V_i^\beta(x))_{(j)}\|^2 \geq \frac{2\lambda_i}{L_i + \beta_i}. \quad (13)$$

Replacing $x = x^k$ for $k \geq 0$, it can be easily seen that, for any $j \in \mathcal{S}_i$ and $i \in [N]$, we have:

$$|x_{(j)}^k|^2 \begin{cases} \geq \frac{2\lambda_i}{L_i + \beta_i}, & \text{if } x_{(j)}^k \neq 0 \text{ and } i \in \xi^k \\ = |x_{(j)}^0|^2, & \text{if } x_{(j)}^k \neq 0 \text{ and } i \notin \xi^k. \end{cases}$$

Further, assume that at some iteration $k > 0$ a change of sequence I^k in expectation occurs. Thus, there is an index $j \in [n]$ (and block i containing j) such that either $\left(x_{(j)}^k = 0 \text{ and } \left(V_i^\beta(x^k)\right)_{(j)} \neq 0\right)$ or $\left(x_{(j)}^k \neq 0 \text{ and } \left(V_i^\beta(x^k)\right)_{(j)} = 0\right)$. Analyzing these cases, we have:

$$\|V_i^\beta(x^k) - x_i^k\|^2 \geq \left\| \left(V_i^\beta(x^k)\right)_{(j)} - x_{(j)}^k \right\|^2 \begin{cases} \geq \frac{2\lambda_i}{L_i + \beta_i} & \text{if } x_{(j)}^k = 0 \\ \geq \frac{2\lambda_i}{L_i + \beta_i} & \text{if } x_{(j)}^k \neq 0 \text{ and } i \in \xi^k \\ = \|x_{(j)}^0\|^2 & \text{if } x_{(j)}^k \neq 0 \text{ and } i \notin \xi^k. \end{cases}$$

Observing that under uniform probabilities we have:

$$\mathbb{E} \left[\frac{\beta_{i_k}}{2} \|x^{k+1} - x^k\|^2 | x^k \right] = \frac{1}{N} \sum_{i=1}^N \frac{\beta_i}{2} \|V_i^\beta(x^k) - x_i^k\|^2,$$

we can conclude that at each change of sequence I^k in expectation we get:

$$\mathbb{E} \left[\frac{\beta_{i_k}}{2} \|x^{k+1} - x^k\|^2 | x^k \right] \geq \frac{1}{N} \min \left\{ \min_{i \in [N]: \lambda_i > 0} \lambda_i \left(1 - \frac{L_i}{L_i + \beta_i} \right), \min_{j \in I^0} \frac{\beta_j}{2} \|x_{(j)}^0\|^2 \right\}.$$

(iv) The first two statements of part (iv) can be proved using the same reasoning as in Theorem 4.3. As the number of changes in expectation of sequence I^k is finite, there exist some $K > 0$ such that for any $k > K$ the sequence I^k does not change in expectation any more and the algorithm (RPAM-IHT) reduces to the classical random proximal alternating minimization [12], for which any limit point x^* of its generated sequence satisfies $\nabla_{I(x^*)} f(x^*) = 0$. Therefore, if the sequence I^k is fixed, then we have for any $k > K$:

$$F(x^{k+1}) + \frac{\beta_{i_k}}{2} \|x_{i_k}^{k+1} - x_{i_k}^k\|^2 \leq F(x^k + U_{i_k} h_{i_k}) + \frac{\beta_{i_k}}{2} \|h_{i_k}\|^2 \quad \forall h_{i_k} \in \mathbb{R}^{n_{i_k}}, i_k \in I^k. \quad (14)$$

On the other hand, denoting with x^* an accumulation point of x^k , taking limit in (14) and using that $\|x^k\|_{0,\lambda} = \|x^*\|_{0,\lambda}$ as $k \rightarrow \infty$, we obtain the following relation:

$$F(x^*) \leq \min_{h_i \in \mathbb{R}^{n_i}} F(x^* + U_i h_i) + \frac{\beta_i}{2} \|h_i\|^2 \quad a.s.$$

for all $i \in [N]$ and thus x^* is the minimizer of the previous right hand side expression. Using the definition of local minimizers from the set $\mathcal{L}(\beta)$ we conclude that any limit point of the sequence x^k belongs to this set, which proves our statement. ■

Note that our results are more general than the ones in [17], where the author proves linear convergence for algorithm (IHTA) under more restricted assumptions on the objective function. Moreover, our proofs for convergence are completely different from those in this paper [17]. It is also important to note that the classical results for any iterative algorithm used for solving nonconvex problems usually state global convergence to stationary points, while for our algorithms we were able to prove global convergence to local minima of our nonconvex and NP-hard problem (1). Moreover, if $\lambda_i = 0$ for all $i \in [N]$, then the optimization problem (1) becomes convex and we see that our convergence results cover also this setting.

V. RATE OF CONVERGENCE ANALYSIS

In this section we analyze the rate of convergence in probability of coordinate descent algorithms (RC-IHT) and (RPAM-IHT) under the strong convexity assumption on f with constant σ .

A. Rate of convergence in probability of (RC-IHT)

We now derive the complexity estimates in probability for the random coordinate descent algorithm (RC-IHT). First, let us recall that in [27] Nesterov provided complexity estimates of a random coordinate descent method for convex optimization problems. In the strongly convex case, he showed that the rate of convergence in expectation is given by:

$$\mathbb{E} [f(x^k) - f(x^*)] \leq (1 - \theta_1)^k (f(x^0) - f(x^*)),$$

where $\theta_1 = \frac{\sigma}{\sum_i L_i}$ and $x^* = \arg \min_{x \in \mathbb{R}^n} f(x)$. Using the strong convexity of f we have:

$$\mathbb{E} [\|x^k - x^*\|] \leq (1 - \theta_1)^{k/2} \sqrt{\frac{2}{\sigma} (f(x^0) - f(x^*))}. \quad (15)$$

Thus, in order to attain $\mathbb{E} [\|x^k - x^*\|] \leq \epsilon$, Nesterov's method requires a number of iterations:

$$k \geq \frac{2}{\theta_1} \log \left(\frac{1}{\epsilon} \sqrt{\frac{2 (f(x^0) - f(x^*))}{\sigma}} \right). \quad (16)$$

We use this bound in our complexity analysis given below. First, we introduce the constant:

$$\gamma = \min \left\{ \left| |s_{M_i}(z)_{(j)}| - \sqrt{2\lambda_i/M_i} \right| : i \in [N], \lambda_i > 0, j \in \mathcal{S}_i, z \in \mathcal{T}_f \right\}.$$

Note that due to the strong convexity of function f , the set of basic local minimizers \mathcal{T}_f is finite. Therefore, with a similar argument as in [17], choosing randomly the constants $M_i >$

L_i we can claim that $\gamma > 0$ almost surely. Our approach for proving linear convergence for random coordinate descent algorithm (RC-IHT) is in a way similar to the one for algorithm (IHTA) in [17]. However, our results are more general in the sense that they allow us to show convergence for a random method which requires a different approach (e.g. proof in probability) and introduction of some new concepts (e.g. change of support in expectation). Last, but not least important, the complexity per iteration of our method is much lower.

From Theorem 4.3 there is a finite number of changes of sequence I^k in expectation. We denote with $k_1, k_2, \dots, k_\kappa$ the iterations at which a change of I^k in expectation occurs.

Theorem 5.1: Let x^k be the sequence generated by algorithm (RC-IHT) for solving optimization problem (1), with f a strongly convex function and $M_i > L_i$ chosen randomly. Denote with κ the number of changes in expectation of I^k as $k \rightarrow \infty$. Also let x^* be the convergence point of x^k in expectation (see (15)) and $\rho > 0$ be some confidence level. Then, we have:

(i) Considering δ as given in Lemma 4.3 (iii), then the number of changes in expectation κ of I^k is at most $\left\lfloor \frac{\mathbb{E}[F(x^0) - F(x^*)]}{\delta} \right\rfloor$.

(ii) The sequence x^k satisfies $\mathbb{P}(F(x^k) - F(x^*) \leq \epsilon) \geq 1 - \rho$ for all $k \geq \log \frac{\tilde{\omega}}{\epsilon}$, where $\tilde{\omega} = 2^{\omega + \frac{1}{\theta_1} \log \frac{\delta}{\epsilon \rho}}$, with $\omega = \left\{ \max_{t \in \mathbb{R}} \alpha t - \beta t^2 : 0 \leq t \leq \left\lfloor \frac{\mathbb{E}[F(x^0) - F(x^*)]}{\delta} \right\rfloor \right\}$, $\beta = \frac{\delta}{4(F(x^0) - F^*)}$ and $\alpha = \frac{1}{\theta_1} \left(\log [2(F(x^0) - F^*)] + 2 \log \frac{2N(L_f + M)}{\sqrt{\sigma \gamma M}} + \frac{\delta}{4(F(x^0) - F^*)} \right)$.

Proof: (i) From Lemma 4.2(ii) and Lemma 4.3(iii) it can be easily seen that:

$$\delta \leq \mathbb{E} \left[\frac{M_{i_{k_p}} - L_{i_{k_p}}}{2} \|x^{k_p+1} - x^{k_p}\|^2 \middle| x^{k_p} \right] \stackrel{(11)}{\leq} F(x^{k_p}) - \mathbb{E}[F(x^{k_p+1}) | x^{k_p}].$$

Taking expectation in this relation w.r.t. the entire history ξ^{k_p} we get the bound: $\delta \leq \mathbb{E}[F(x^{k_p}) - F(x^{k_p+1})]$. Further, summing up over the entire history we have:

$$\delta \kappa \leq \mathbb{E}[F(x^{k_1}) - F(x^{k_\kappa+1})] \leq \mathbb{E}[F(x^0) - F(x^*)].$$

(ii) In order to state the iteration complexity in probability of algorithm (RC-IHT), we require a bound on the number of iterations performed between two changes in expectation of I^k . Recall that for any $p \in [\kappa]$, at iteration $k_p + 1$, the set I^{k_p} changes in expectation, i.e.:

$$\mathbb{E}[|I^{k_p}/I^{k_p+1}| + |I^{k_p+1}/I^{k_p}| | x^{k_p}] > 0,$$

which implies

$$\mathbb{P}(|I^{k_p}/I^{k_p+1}| + |I^{k_p+1}/I^{k_p}| > 0) = \mathbb{P}(I^{k_p} \neq I^{k_p+1}) \geq \frac{1}{N}$$

and furthermore

$$\mathbb{P}(|I^{k_p}/I^{k_p+1}| + |I^{k_p+1}/I^{k_p}| = 0) = \mathbb{P}(I^{k_p} = I^{k_p+1}) \leq \frac{N-1}{N}. \quad (17)$$

Let p be an arbitrary integer from $[\kappa]$. Denote $\hat{x}^* = \arg \min_{x \in S_{I^{k_p}}} f(x)$ and $\hat{f}^* = \mathbb{E}[f(\hat{x}^*)]$. Assume that we have:

$$k_p - k_{p-1} > \frac{1}{\theta_1} \left(\log [2(F(x^0) - F^* - (p-1)\delta)] + 2 \log \frac{2N(L_f + \underline{M})}{\sqrt{\sigma}\gamma\underline{M}} \right). \quad (18)$$

From Lemma 4.2(ii) and Lemma 4.3(iii) we have:

$$f^{k_{p-1}+1} - \hat{f}^* \leq \mathbb{E}[F(x^{k_{p-1}+1})] - \mathbb{E}[F(\hat{x}^*)] \leq F(x^0) - F^* - (p-1)\delta.$$

Thus, we claim that (18) implies:

$$\begin{aligned} k_p - k_{p-1} &> \frac{2}{\theta_1} \log \frac{2\sqrt{2(f^{k_{p-1}+1} - \hat{f}^*)N(L_f + \underline{M})}}{\sqrt{\sigma}\gamma\underline{M}} \\ &\geq \frac{2}{\theta_1} \log \frac{\sqrt{2N(f^{k_{p-1}+1} - \hat{f}^*)(L_f + \underline{M})}}{\sqrt{\sigma}\gamma\underline{M}(\sqrt{N} - \sqrt{N-1})}. \end{aligned} \quad (19)$$

We show that under relation (19), the probability (17) does not hold. First, we see that between two changes in expectation of I^k , i.e. $k \in [k_{p-1}+1, k_p]$, the algorithm (RC-IHT) is equivalent with the Nesterov's coordinate descent method. Therefore, the method exhibits the rate of convergence (15), which in our case is given in the following terms:

$$\mathbb{E}[\|x^k - \hat{x}^*\|] \leq (1 - \theta_1)^{(k-k_{p-1}-1)/2} \sqrt{\frac{2}{\sigma}} (f^{k_{p-1}+1} - \hat{f}^*),$$

for all $k \in [k_{p-1}+1, k_p]$. Taking $k = k_p$, if we apply the complexity estimate (16) and use the bound (19), we get:

$$\mathbb{E}[\|x^{k_p} - \hat{x}^*\|] \leq (1 - \theta_1)^{(k_p-k_{p-1}-1)/2} \sqrt{\frac{2}{\sigma}} (f^{k_{p-1}+1} - \hat{f}^*) < \frac{\gamma\underline{M}}{L_f + \underline{M}} \left(1 - \sqrt{1 - \frac{1}{N}}\right).$$

From the Markov inequality, it can be easily seen that we have:

$$\mathbb{P}\left(\|x^{k_p} - \hat{x}^*\| < \frac{\gamma\underline{M}}{L_f + \underline{M}}\right) = 1 - \mathbb{P}\left(\|x^{k_p} - \hat{x}^*\| \geq \frac{\gamma\underline{M}}{L_f + \underline{M}}\right) > \sqrt{1 - \frac{1}{N}}.$$

Further, let $i \in [N]$ such that $\lambda_i > 0$ and $j \in S_i$. From definition of parameter γ and the basic Lipschitz continuous gradient inequality we see that the event $\|x^{k_p} - \hat{x}^*\| \leq \frac{\gamma\underline{M}}{L_f + \underline{M}}$ implies:

$$\begin{aligned} \left| |s_{M_i}(x^{k_p})_{(j)}| - |s_{M_i}(\hat{x}^*)_{(j)}| \right| &\leq |x_{(j)}^{k_p} - \hat{x}_{(j)}^*| + \frac{1}{M_i} |\nabla_{(j)} f(x^{k_p}) - \nabla_{(j)} f(\hat{x}^*)| \\ &\leq (1 + L_f/\underline{M}) \|x^{k_p} - \hat{x}^*\| < \gamma \leq \left| |s_{M_i}(\hat{x}^*)_{(j)}| - \sqrt{2\lambda_i/M_i} \right|. \end{aligned}$$

The first and the last terms from the above inequality imply further that:

$$\begin{cases} |s_{M_i}(x^{k_p})_{(j)}| > \sqrt{2\lambda_i/M_i}, & \text{if } |s_{M_i}(\hat{x}^*)_{(j)}| > \sqrt{2\lambda_i/M_i} \\ |s_{M_i}(x^{k_p})_{(j)}| < \sqrt{2\lambda_i/M_i}, & \text{if } |s_{M_i}(\hat{x}^*)_{(j)}| < \sqrt{2\lambda_i/M_i}, \end{cases}$$

or $I^{k_p+1} = \hat{I}^* = \{j \in [n] : j \in \mathcal{S}_i, \lambda_i = 0\} \cup \{j \in [n] : j \in \mathcal{S}_i, \lambda_i > 0, |s_{M_i}(\hat{x}^*)_{(j)}| > \sqrt{2\lambda_i/L_i}\}.$

In conclusion, assuming (19) we have:

$$\mathbb{P}(I^{k_p+1} = \hat{I}^*) > \sqrt{1 - \frac{1}{N}}.$$

Applying the same procedure for iteration k_p we obtain:

$$\mathbb{P}(I^{k_p} = \hat{I}^*) > \sqrt{1 - \frac{1}{N}}.$$

Considering events $\{I^{k_p} = \hat{I}^*\}$ and $\{I^{k_p+1} = \hat{I}^*\}$ to be independent, we have:

$$\mathbb{P}\left(\{I^{k_p+1} = \hat{I}^*\} \cap \{I^{k_p} = \hat{I}^*\}\right) = \mathbb{P}(I^{k_p+1} = I^{k_p}) > \frac{N-1}{N},$$

which contradicts the assumption $P(I^{k_p} = I^{k_p+1}) \leq \frac{N-1}{N}$. Therefore, between two changes of sequence I^k the number of iterations is bounded by:

$$k_p - k_{p-1} \leq \frac{1}{\theta_1} \left(\log [2(F(x^0) - F^* - (p-1)\delta)] + 2 \log \frac{2N(L_f + \underline{M})}{\sqrt{\sigma}\gamma\underline{M}} \right).$$

If we use the inequality $\log(1-x) \leq -x$ for any $x \in (0, 1)$, and denote with k_κ the number of iterations until the last change in expectation of sequence I^k , we have:

$$\begin{aligned} k_\kappa &\leq \sum_{p=1}^{\kappa} \frac{1}{\theta_1} \left(\log [2(F(x^0) - F^*)] - \frac{(p-1)\delta}{2(F(x^0) - F^*)} + 2 \log \frac{2N(L_f + \underline{M})}{\sqrt{\sigma}\gamma\underline{M}} \right) \\ &= \underbrace{\kappa \frac{1}{\theta_1} \left(\log [2(F(x^0) - F^*)] + 2 \log \frac{2N(L_f + \underline{M})}{\sqrt{\sigma}\gamma\underline{M}} \right)}_{\alpha} + \underbrace{\kappa^2 \frac{\delta}{4(F(x^0) - F^*)}}_{\beta} \\ &\leq \omega. \end{aligned}$$

In order to reach some ϵ -local minimum in probability with some confidence parameter ρ , the Algorithm (RC-IHT) has to additionally perform another

$$\frac{1}{\theta_1} \log \frac{f^{k_\kappa+1} - \phi^*}{\epsilon\rho}$$

iterations, where $\phi^* = \lim_{k \rightarrow \infty} f^k = \mathbb{E} \left[\min_{x \in S_{I^{k_{\kappa}+1}}} f(x) \right]$. Taking into account that the iteration κ is the largest possible integer at which the sequence I^k could change, we can bound $f^{k_{\kappa}+1} - \phi^* \leq \delta$ and obtain:

$$\frac{1}{\theta_1} \log \frac{f^{k_{\kappa}+1} - \phi^*}{\epsilon \rho} \leq \frac{1}{\theta_1} \log \frac{\delta}{\epsilon \rho}.$$

Adding up all the quantities, we obtain our result. ■

B. Rate of convergence in probability of (RPAM-IHT)

In this section we prove the linear convergence in probability of the random coordinate descent algorithm (RPAM-IHT) under strong convexity assumption on function f and for the scalar case, i.e. we assume $n_i = 1$ for all i . Note that, for algorithm (RPAM-IHT) the scalar case is the most practical since it requires solving a simple unidimensional convex problem, while for $n_i > 1$ it requires the solution of an NP-hard problem at each iteration. First, let us recall that complexity results for alternating minimization methods under convexity and Lipschitz gradient assumptions have been developed e.g. in [12], where a sublinear rate has been derived for a general class of coordinate descent methods. Using a similar reasoning as in [12], it can be derived immediately that the randomized version of alternating minimization method in the strongly convex case presents a linear rate of convergence in expectation of the form:

$$\mathbb{E}[f(x^k) - f(x^*)] \leq (1 - \theta_2)^k (f(x^0) - f(x^*)),$$

where $\theta_2 \in (0, 1)$. Using the strong convexity property for f we have:

$$\mathbb{E} [\|x^k - x^*\|] \leq (1 - \theta_2)^{k/2} \sqrt{\frac{2}{\sigma} (f(x^0) - f(x^*))}. \quad (20)$$

For attaining an ϵ -suboptimality this algorithm has to perform the following number of iterations:

$$k \geq \frac{2}{\theta_2} \log \frac{1}{\epsilon} \sqrt{\frac{2(f(x^0) - f(x^*))}{\sigma}}. \quad (21)$$

In order to derive the probabilistic complexity of algorithm (RPAM-IHT), we first state an auxiliary result.

Lemma 5.2: For any $x, y \in \mathbb{R}^n$ the following relation holds:

$$|\Delta^i(x) - \Delta^i(y)| \leq (\|\nabla f(y - U_i y_i)\| + \|\nabla f(v^i(y))\| + \beta_i \|y_i\|) \|x - y\| + \frac{L_f + \beta_i}{2} \|x - y\|^2.$$

Proof: First we use the triangle inequality to derive the following relation:

$$\begin{aligned} |\Delta^i(x) - \Delta^i(y)| &\leq |f(x - U_i x_i) - f(y - U_i y_i) + f(v^i(y)) - f(v^i(x))| \\ &\quad + \left| \frac{\beta_i}{2} \|v^i(y)_i - y_i\|^2 - \frac{\beta_i}{2} \|v^i(x)_i - x_i\|^2 + \left| \frac{\beta_i}{2} \|x_i\|^2 - \frac{\beta_i}{2} \|y_i\|^2 \right| \right|. \end{aligned}$$

For simplicity, we denote:

$$\begin{aligned} \delta_{1i}(x, y) &= f(x - U_i x_i) - f(y - U_i y_i) + f(v^i(y)) - f(v^i(x)) \\ &\quad + \frac{\beta_i}{2} \|v^i(y)_i - y_i\|^2 - \frac{\beta_i}{2} \|v^i(x)_i - x_i\|^2 \\ \delta_{2i}(x, y) &= \frac{\beta_i}{2} \|x_i\|^2 - \frac{\beta_i}{2} \|y_i\|^2. \end{aligned}$$

In order to bound $\Delta^i(x) - \Delta^i(y)$, it is sufficient to find upper bounds on $|\delta_{1i}(x, y)|$ and $|\delta_{2i}(x, y)|$. For a bound on $|\delta_{1i}(x, y)|$ we use $|\delta_{1i}(x, y)| = \max\{\delta_{1i}(x, y), -\delta_{1i}(x, y)\}$. Using the optimality conditions for the map $v^i(x)$ and convexity of f we obtain:

$$\begin{aligned} f(v^i(x)) &\geq f(v^i(y)) + \langle \nabla f(v^i(y)), v^i(x) - v^i(y) \rangle \\ &= f(v^i(y)) + \langle \nabla f(v^i(y)), x - y \rangle + \langle \nabla_i f(v^i(y)), (v^i(x)_i - x_i) - (v^i(y)_i - y_i) \rangle \\ &= f(v^i(y)) + \langle \nabla f(v^i(y)), x - y \rangle - \beta_i \langle v^i(y)_i - y_i, (v^i(x)_i - x_i) - (v^i(y)_i - y_i) \rangle \\ &\geq f(v^i(y)) + \frac{\beta_i}{2} \|v^i(y)_i - y_i\|^2 - \frac{\beta_i}{2} \|v^i(x)_i - x_i\|^2 - \|\nabla f(v^i(y))\| \|x - y\|, \end{aligned} \quad (22)$$

where in the last inequality we used the Cauchy-Schwartz inequality. On the other hand, from the global Lipschitz continuous gradient inequality we get:

$$f(x - U_i x_i) \leq f(y - U_i y_i) + \|\nabla f(y - U_i y_i)\| \|x - y\| + \frac{L_f}{2} \|x - y\|^2. \quad (23)$$

From (22) and (23) we obtain:

$$\delta_{1i}(x, y) \leq (\|\nabla f(y - U_i y_i)\| + \|\nabla f(v^i(y))\|) \|x - y\| + \frac{L_f}{2} \|x - y\|^2. \quad (24)$$

In order to obtain a bound on $-\delta_{1i}(x, y)$ we observe that:

$$\begin{aligned} f(v^i(x)) + \frac{\beta_i}{2} \|v^i(x)_i - x_i\|^2 - f(v^i(y)) - \frac{\beta_i}{2} \|v^i(y)_i - y_i\|^2 \\ \leq f(x + U_i(v^i(y)_i - y_i)) - f(v^i(y)) \leq \|\nabla f(v^i(y))\| \|x - y\| + \frac{L_f}{2} \|x - y\|^2, \end{aligned} \quad (25)$$

where in the last inequality we used the Lipschitz gradient relation and Cauchy-Schwartz inequality. Also, from the convexity of f and the Cauchy-Schwartz inequality we get:

$$f(x - U_i x_i) \geq f(y - U_i y_i) - \|\nabla f(y - U_i y_i)\| \|x - y\|. \quad (26)$$

Combining now the bounds (25) and (26) we obtain:

$$-\delta_{1i}(x, y) \leq (\|\nabla f(y - U_i y_i)\| + \|\nabla f(v^i(y))\|) \|x - y\| + \frac{L_f}{2} \|x - y\|^2. \quad (27)$$

Therefore, from (24) and (27) we obtain a bound on $\delta_{1i}(x, y)$:

$$|\delta_{1i}(x, y)| \leq (\|\nabla f(y - U_i y_i)\| + \|\nabla f(v^i(y))\|) \|x - y\| + \frac{L_f}{2} \|x - y\|^2. \quad (28)$$

Regarding the second quantity $\delta_{2i}(x, y)$, we observe that:

$$\begin{aligned} |\delta_{2i}(x, y)| &= \frac{\beta_i}{2} \left| |x_i| + |y_i| \right| \left| |x_i| - |y_i| \right| = \frac{\beta_i}{2} \left| |x_i| - |y_i| + 2|y_i| \right| \left| |x_i| - |y_i| \right| \\ &\leq \frac{\beta_i}{2} (\|x - y\| + 2\|y_i\|) \|x - y\|. \end{aligned} \quad (29)$$

From the upper bounds on $|\delta_{1i}(x, y)|$ and $|\delta_{2i}(x, y)|$, (28) and (29), we obtained our result. ■

We use the same notations as in Theorem 5.1. Additionally, we use the following notations:

$$\tilde{\gamma} = \max_{z \in \mathcal{T}_f} \max_{i \in [n]} \|\nabla f(z - U_i z_i)\| + \|\nabla f(v^i(z))\| + \beta_i \|z_i\|, \quad \tilde{\alpha} = \min_{z \in \mathcal{T}_f} \min_{i \in [n]} |\Delta^i(z) - \lambda_i|.$$

Since the cardinality of \mathcal{T}_f is finite, then there is a finite number of possible values for $|\Delta^i(z) - \lambda_i|$. Therefore, we obtain that $\tilde{\alpha} = 0$ for a finite number of values of parameter β_i . In conclusion, choosing randomly at an initialization stage the parameters $\beta_i > 0$, we have that $\tilde{\alpha} > 0$ almost sure. Further, we state the rate of convergence in probability for algorithm (RPAM-IHT). We use the notation k_p for the iterations when a change in expectation of I^k occurs, as in the previous subsection.

Theorem 5.3: Let x^k be the sequence generated by algorithm (RPAM-IHT) using uniform probabilities and $\beta_i > 0$ chosen randomly. Denote with κ the number of changes in expectation of I^k as $k \rightarrow \infty$. Let x^* be the convergence point of x^k in expectation (see (20)) and $\rho > 0$ be some confidence level. Under the strong convexity of f and $n_i = 1$ we have the results:

(i) The number of changes in expectation κ of I^k is bounded by $\frac{\mathbb{E}[F(x^0) - F(x^*)]}{\delta}$, where δ is specified in Lemma 4.5 (iii).

(ii) The sequence x^k satisfies $\mathbb{P}(F(x^k) - F(x^*) \leq \epsilon) \geq 1 - \rho$ for $k \geq \log \frac{\tilde{\omega}}{\epsilon}$, where $\tilde{\omega} = 2^{\omega + \frac{2}{\theta_2} \log \frac{\delta}{\epsilon \rho}}$, with $\omega = \left\{ \max_{t \in \mathbb{R}} \alpha t - \beta t^2 : 0 \leq t \leq \left\lfloor \frac{\mathbb{E}[F(x^0) - F(x^*)]}{\delta} \right\rfloor \right\}$, $\beta = \frac{\delta}{4(F(x^0) - F^*)}$, $\alpha = \frac{1}{\theta_2} \left(\log [2(F(x^0) - F^*)] + 2 \log \frac{2N}{\sqrt{\sigma} \xi} + \frac{\delta}{4(F(x^0) - F^*)} \right)$ and $\xi = \sqrt{\frac{\tilde{\gamma}^2}{(L_f + \beta)^2} + \frac{2\tilde{\alpha}}{L_f + \beta}} - \frac{\tilde{\gamma}}{L_f + \beta}$.

Proof: The first part of the proof (i.e. part (i)) can be easily proved using similar arguments as in the proof of Theorem 5.1 (i).

(ii) In order to establish the iteration complexity in probability of algorithm (RPAM-IHT), we require a bound on the number of iterations performed between two changes in expectation I^k . Recall that for any $p \in [\kappa]$, at iteration $k_p + 1$, there is a change in expectation of I^{k_p} , i.e.

$$\mathbb{E}[|I^{k_p}/I^{k_p+1}| + |I^{k_p+1}/I^{k_p}| | x^{k_p}] > 0,$$

which implies that

$$\mathbb{P}(|I^{k_p}/I^{k_p+1}| + |I^{k_p+1}/I^{k_p}| > 0) = \mathbb{P}(I^{k_p} \neq I^{k_p+1}) \geq \frac{1}{N}$$

and furthermore

$$\mathbb{P}(|I^{k_p}/I^{k_p+1}| + |I^{k_p+1}/I^{k_p}| = 0) = \mathbb{P}(I^{k_p} = I^{k_p+1}) \leq \frac{N-1}{N}. \quad (30)$$

Let p be an arbitrary integer from $[\kappa]$. Denote $\hat{x}^* = \arg \min_{x \in S_{I^{k_p}}} f(x)$ and $\hat{f}^* = \mathbb{E}[f(\hat{x}^*)]$. Assume that we have:

$$k_p - k_{p-1} > \frac{1}{\theta_2} \left(\log [2(F(x^0) - F^* - (p-1)\delta)] + 2 \log \frac{2N}{\sqrt{\sigma}\xi} \right). \quad (31)$$

From Lemma 4.4(ii) and Lemma 4.5(iii) we have:

$$f^{k_{p-1}+1} - \hat{f}^* \leq \mathbb{E}[F(x^{k_{p-1}+1})] - \mathbb{E}[F(\hat{x}^*)] \leq F(x^0) - F^* - (p-1)\delta,$$

so that we can claim that (31) implies

$$k_p - k_{p-1} > \frac{2}{\theta_2} \log \frac{2\sqrt{2(f^{k_{p-1}+1} - \hat{f}^*)N}}{\sqrt{\sigma}\xi} \geq \frac{2}{\theta_2} \log \frac{\sqrt{2N(f^{k_{p-1}+1} - \hat{f}^*)}}{\sqrt{\sigma}\xi(\sqrt{N} - \sqrt{N-1})}. \quad (32)$$

We show that under relation (32), the probability (30) does not hold. First, we observe that between two changes in expectation of I^k , i.e. $k \in [k_{p-1} + 1, k_p]$, the algorithm (RPAM-IHT) is equivalent with the randomized version of the alternating minimization method [12]. Therefore, the method has linear rate of convergence (20), which in our case is given in the following terms:

$$\mathbb{E}[\|x^k - \hat{x}^*\|] \leq (1 - \theta_2)^{(k - k_{p-1} - 1)/2} \sqrt{\frac{2}{\sigma}} (f^{k_{p-1}+1} - \hat{f}^*),$$

for all $k \in [k_{p-1} + 1, k_p]$. Taking $k = k_p$, if we apply the complexity estimate (21) and use the bound (32), we obtain:

$$\mathbb{E}[\|x^{k_p} - \hat{x}^*\|] \leq (1 - \theta_1)^{(k_p - k_{p-1} - 1)/2} \sqrt{\frac{2}{\sigma}} (f^{k_{p-1}+1} - \hat{f}^*) < \xi \left(1 - \sqrt{\frac{N-1}{N}} \right).$$

From the Markov inequality, it can be easily seen that we have:

$$\mathbb{P}(\|x^{k_p} - \hat{x}^*\| < \xi) = 1 - \mathbb{P}(\|x^{k_p} - \hat{x}^*\| \geq \xi) > \sqrt{1 - \frac{1}{N}}.$$

Let $i \in [N]$ such that $\lambda_i > 0$. From Lemma 5.2 and definition of parameter ξ we see that the event $\|x^{k_p} - \hat{x}^*\| < \xi$ implies:

$$|\Delta^i(x^{k_p}) - \Delta^i(\hat{x}^*)| \leq \tilde{\gamma}\|x^{k_p} - \hat{x}^*\| + \frac{L_f + \bar{\beta}}{2}\|x^{k_p} - \hat{x}^*\|^2 < \tilde{\alpha} \leq |\Delta^i(\hat{x}^*) - \lambda_i|.$$

The first and the last terms from the above inequality further imply:

$$\begin{cases} |\Delta^i(x^{k_p})| > \lambda_i, & \text{if } |\Delta^i(\hat{x}^*)| > \lambda_i \\ |\Delta^i(x^{k_p})| < \lambda_i, & \text{if } |\Delta^i(\hat{x}^*)| < \lambda_i, \end{cases}$$

or $I^{k_p+1} = \hat{I}^* = \{j \in [n] : \lambda_j = 0\} \cup \{i \in [n] : \lambda_i > 0, |\Delta^i(\hat{x}^*)| > \lambda_i\}$. Using the same reasoning as in Theorem 5.1, we obtain a contradiction with the assumption $\mathbb{P}(I^{k_p} = I^{k_p+1}) \leq \frac{N-1}{N}$ and thus with the bound (32). The estimation of the number of iterations for attaining an ϵ -approximate minimizer with probability $1 - \rho$ follows very similar lines as in the proof of Theorem 5.1 and thus we omit the derivations. \blacksquare

VI. NUMERICAL EXPERIMENTS

In this section we analyze the practical performances of our new algorithms (RC-IHT) and (RPAM-IHT) and compare them with that of algorithm (IHTA) [17]. First we perform several numerical tests on sparse learning problems with randomly generated data. Then, we apply our algorithms for solving a robust state estimation problem with perturbed data, arising e.g. in power systems (for simulations we use IEEE 14 bus test case).

A. Random data experiments on sparse learning

Sparse learning represents a collection of learning methods which seek a tradeoff between some goodness-of-fit measure and sparsity of the result, the latter property allowing better interpretability. One of the models widely used in machine learning and statistics is the linear model (least squares setting). Thus, in the first set of tests we consider sparse linear formulation:

$$\min_{x \in \mathbb{R}^n} F(x) \quad \left(= \frac{1}{2} \|Ax - b\|^2 + \lambda \|x\|_0 \right),$$

where $A \in \mathbb{R}^{m \times n}$ and $\lambda > 0$. We analyze the practical efficiency of our algorithms in terms of the probability of reaching a global optimal point. Due to difficulty of finding the global solution of this problem, we consider a small model $m = 6$ and $n = 12$. For each penalty parameter λ , ranging from small values (0.01) to large values (2), we ran algorithms (RC-IHT), (RPAM-IHT) and (IHTA) [17] from 100 randomly generated (with random support) initial vectors. The numbers of runs out of 100 in which each method found the global optimum is given in Table II. We observe that for all values of λ our algorithms (RC-IHT) and (RPAM-IHT) are able to identify the global optimum with a rate of success superior to algorithm (IHTA) and for extreme values of λ our algorithms perform much better than (IHTA).

TABLE II

NUMBERS OF RUNS OUT OF 100 IN WHICH ALGORITHMS (IHTA), (RC-IHT) AND (RPAM-IHT) FOUND GLOBAL OPTIMUM.

| λ | (IHTA) | (RC-IHT) | (RPAM-IHT) |
|-----------|--------|----------|------------|
| 0.01 | 95 | 96 | 100 |
| 0.07 | 92 | 92 | 100 |
| 0.09 | 43 | 51 | 70 |
| 0.15 | 41 | 47 | 66 |
| 0.35 | 24 | 28 | 27 |
| 0.8 | 36 | 43 | 44 |
| 1.2 | 29 | 29 | 54 |
| 1.8 | 76 | 81 | 91 |
| 2 | 79 | 86 | 97 |

In the second set of experiments we consider the ℓ_2 regularized logistic loss model from machine learning [1]. In this model the relation between the data, represented by a random vector $a \in \mathbb{R}^p$, and its associated label, represented by a random binary variable $y \in \{0, 1\}$, is determined by the conditional probability:

$$P\{y|a; x\} = \frac{e^{y\langle a, x \rangle}}{1 + e^{\langle a, x \rangle}},$$

where x denotes a parameter vector. Then, for a set of n independently drawn data samples $\{(a_i, y_i)\}_{i=1}^n$, the joint likelihood can be written as a function of x . To find the maximum likelihood estimate one should maximize the likelihood function, or equivalently minimize the

negative log-likelihood (the logistic loss):

$$\min_{x \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n \log(1 + e^{\langle a_i, x \rangle}) - y_i \langle a_i, x \rangle.$$

Under the assumption of $p \leq n$ and $A = [a_1, \dots, a_n] \in \mathbb{R}^{m \times n}$ being full rank, it is well known that $f(\cdot)$ is strictly convex. However, there are important applications (e.g. feature selection) where these assumptions are not satisfied and the problem is highly ill-posed. In order to compensate this drawback, the logistic loss is regularized by some penalty term (e.g. ℓ_2 norm $\|x\|_2^2$, see [1], [11]). Furthermore, the penalty term implicitly bounds the length of the minimizer, but does not promote sparse solutions. Therefore, it is desirable to impose an additional sparsity regularizer, such as the ℓ_0 quasinorm. In conclusion our problem to be minimized is given by:

$$\min_{x \in \mathbb{R}^n} F(x) \quad \left(= \frac{1}{n} \sum_{i=1}^n \log(1 + e^{\langle a_i, x \rangle}) - y_i \langle a_i, x \rangle + \frac{\nu}{2} \|x\|^2 + \|x\|_{0,\lambda} \right),$$

where now f is strongly convex with parameter ν . For simulation, data were uniformly random generated and we fixed the parameters $\nu = 1$ and $\lambda = 0.1$. Once an instance of random data has been generated, we ran 10 times our algorithms (RC-IHT) and (RPAM-IHT) and algorithm (IHTA) [17] starting from 10 different initial points. We reported in Table III the best results of each algorithm obtained over all 10 trials, in terms of best function value that has been attained with associated sparsity and number of iterations. In order to report relevant information, we have measured the performance of coordinate descent methods (RC-IHT) and (RPAM-IHT) in terms of full iterations obtained by dividing the number of all iterations by the dimension n . The column F^* denotes the final function value attained by the algorithms, $\|x^*\|_0$ represents the sparsity of the last generated point and *iter (full-iter)* represents the number of iterations (the number of full iterations). Note that our algorithms (RC-IHT) and (RPAM-IHT) have superior performance in comparison with algorithm (IHTA) on the reported instances. We observe that algorithm (RPAM-IHT) performs very few full iterations in order to attain best function value amongst all three algorithms. Moreover, the number of full iterations performed by algorithm (RPAM-IHT) scales up very well with the dimension of the problem.

B. Application to robust state estimation

An important problem in control applications is the estimation of state variables of dynamic systems in noisy environments. There are several formulations of this problem under various

TABLE III
PERFORMANCE OF ALGORITHMS (IHTA), (RC-IHT), (RPAM-IHT)

| $m \backslash n$ | (IHTA) | | | (RC-IHT) | | | (RPAM-IHT) | | |
|------------------|---------|-------------|-------|----------|-------------|-----------|------------|-------------|-----------|
| | F^* | $\ x^*\ _0$ | iter | F^* | $\ x^*\ _0$ | full-iter | F^* | $\ x^*\ _0$ | full-iter |
| 10\100 | 0.90 | 2 | 97 | 0.82 | 1 | 48 | 0.55 | 2 | 12 |
| 30\200 | -14.792 | 45 | 969 | -14.797 | 45 | 813 | -70.50 | 123 | 13 |
| 50\300 | 2.03 | 16 | 2354 | 1.39 | 12 | 3709 | -1.16 | 25 | 15 |
| 70\500 | 14.69 | 148 | 4324 | 15.57 | 156 | 3810 | 0.65 | 1 | 11 |
| 100\1500 | 30.54 | 329 | 7367 | 25.58 | 375 | 7205 | 0.59 | 4 | 9 |
| 150\2000 | 32.22 | 408 | 15097 | 30.84 | 454 | 14230 | -3.06 | 74 | 14 |

assumptions on noise and perturbations. Suppose we gather the measurements y that obey the nonlinear model $y = h(x) + w$, where x is the vector of system states, w is a disturbance term and h is a vector of m non-linear functions. The error vector w is assumed to have zero mean and a known covariance matrix Σ . Typically, such models are iteratively linearized via the Gauss-Newton method, or by resorting to the so called DC approximation [9], [20]. Thus, after the linearization step, one arrives at the following common linear model [14]:

$$y = Hx + w,$$

where $H \in \mathbb{R}^{m \times n}$ is the Jacobian matrix. In most practical applications we are confronted with bad data and corrupted measurements. For example, in robust state estimation for power systems error sources such as time skews, instrument/communication failures, infrequent instrument calibration and parameter uncertainty can yield grossly corrupted SCADA measurements. Moreover, in the smart grid context, bad data are not simply unintentional metering faults, but can also take the form of malicious data injections [14], [15]. Therefore, a refined model is necessarily defined as:

$$y = Hx + v + w,$$

where v is a vector with $v_{(j)} \neq 0$ only if $y_{(j)}$ is a bad datum [14]. Recovering both x and v reveals the state and identifies faulty measurements. As this recovery is almost impossible, constraining the sparsity of v allows us to attain an approximate recovery. There are previous

attempts to formulate this problem as a sparsity constrained one [14]:

$$\min_{x \in \mathbb{R}^n, v \in \mathbb{R}^m} \left\{ \frac{1}{2} \|y - Hx - v\|^2 \text{ s.t. } \|v\|_0 \leq \kappa \right\}.$$

A typical reformulation of this problem (see e.g. [4]) is the ℓ_0 regularized formulation for a given penalty parameter $\tilde{\lambda} > 0$:

$$\min_{x \in \mathbb{R}^n, v \in \mathbb{R}^m} \frac{1}{2} \|y - Hx - v\|^2 + \tilde{\lambda} \|v\|_0.$$

It can be easily seen that this model is a particular case of the general optimization problem (1):

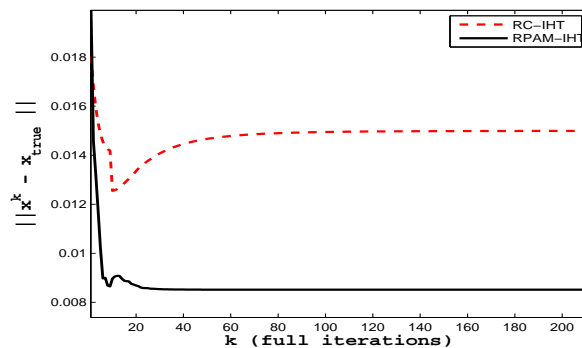
$$\min_{z \in \mathbb{R}^{n+m}} \frac{1}{2} \|y - Az\|^2 + \|z\|_{0,\lambda},$$

where $z = [x^T \ v^T]^T$, $A = [H \ I_m]$ and $\lambda \in \mathbb{R}^{m+n}$, with $\lambda_{1:n} = 0$ and $\lambda_{n+1:n+m} = \tilde{\lambda}$. For simulations we have chosen the state estimation of IEEE 14 bus test case (see [14] for details). In this application, the measurements y are given by corrupted complex values of bus voltages and line currents, and the states are the real and imaginary parts of bus voltages. Also, we created the matrix H following the settings given in [14]. The nonzero weights vector is taken as $\lambda_{n+1:n+m} = \tilde{\lambda}$, with $\tilde{\lambda}$ obtained by tuning. At the initialization stage, we have generated a uniform random state vector w and a sparse bad data vector v using a normal distribution of mean 5. In these settings, we ran algorithm (RPAM-IHT) using a tuned $\tilde{\lambda}$ from which we fully recovered the bad data support that we have generated. In Figure 1 we have plotted the evolution of error $\|x^k - x^*\|$ along full iterations corresponding to algorithms (RC-IHT) and (RPAM-IHT). We observed that both algorithms are able to recover the bad data support, but the algorithm (RPAM-IHT) is better than algorithm (RC-IHT) regarding the estimation of the state.

VII. CONCLUSIONS

In this paper we have derived efficient algorithms for solving ℓ_0 regularized optimization problems, where the objective function is composed of a smooth convex function and the ℓ_0 regularization. We have analyzed necessary optimality conditions for this nonconvex problem which lead to the separation of the local minima into two restricted classes. Based on these restricted classes of local minima we have devised random coordinate descent type methods for solving our problem. We have proved that for each algorithm any limit point is a local minimum from one of these restricted classes of local minimizers. Under the strong convexity assumption we have also proved linear convergence in probability for both methods. Finally,

Fig. 1. Behaviour of error $\|x^k - x^*\|$ for algorithms (RC-IHT) and (RPAM-IHT) along full iterations.



we have performed several numerical experiments which showed the superior behaviour of our methods in comparison with the usual iterative hard thresholding algorithm.

REFERENCES

- [1] S. Bahmani, B. Raj, and P. T. Boufounos, *Greedy sparsity-constrained optimization*, Journal of Machine Learning Research, 14(3), 807841, 2013.
- [2] A. Beck, Y.C. Eldar, *Sparsity Constrained Nonlinear Optimization: Optimality Conditions and Algorithms*, SIAM Journal on Optimization, 23(3), 14801509, 2013.
- [3] C.M. Bishop. *Pattern Recognition and Machine Learning*, Springer, 2007.
- [4] T. Blumensath and M. E. Davies, *Iterative thresholding for sparse approximations*, Journal of Fourier Analysis and Applications, 14, 629-654, 2008.
- [5] T. Blumensath and M. E. Davies, *Iterative hard thresholding for compressed sensing*, Applications of Computational Harmonic Analysis, 27(3), 265-274, 2009.
- [6] E. J. Candes and T. Tao, *Near-optimal signal recovery from random projections: universal encoding strategies*, IEEE Transactions on Information Theory, 52, 5406-5425, 2004.
- [7] E. J. Candes and T. Tao, *Decoding by linear programming*. IEEE Transactions Information Theory, 51, 4203-4215, 2004.
- [8] M. Carlván and L. Blanc-Feraud, *Two constrained formulations for deblurring Poisson noisy images*, IEEE ICIP, 2011.
- [9] D. Duan, L. Yang, and L. L. Scharf, *Phasor state estimation from PMU measurements with bad data*, Proceedings IEEE Workshop on Computational Advances in Multi-Sensor Adaptive Processes, 2011.
- [10] N. Gillis, *Sparse and Unique Nonnegative Matrix Factorization Trough Data Preprocessing*, Journal of Machine Learning Research, 13, 3349-3386, 2012.
- [11] T. Hastie, R. Tibshirani, and J. Friedman, *The elements of statistical learning: data mining, inference and prediction*, Springer Verlag, 2009.
- [12] M. Hong, X. Wang, M. Razaviyayn, Z.-Q. Luo, *Iteration Complexity Analysis of Block Coordinate Descent Methods*, <http://arxiv.org/abs/1310.6957>, 2013.

- [13] M. Journe, Y. Nesterov, P. Richtik and R. Sepulchre, *Generalized power method for sparse principal component analysis*, Journal of Machine Learning Research, 11, 517553, 2010.
- [14] V. Kekatos and G. Giannakis, *Distributed Robust Power System State Estimation*, IEEE Transactions on Power Systems, 28(2), 1617-1626, 2013.
- [15] O. Kosut, L. Jia, J. Thomas, and L. Tong, *Malicious data attacks on the smart grid*, IEEE Transactions on Smart Grids, 2(4), 645658, 2011.
- [16] F. Lin, M. Fardad, and M. R. Jovanovic, *Design of optimal sparse feedback gains via the alternating direction method of multipliers*, IEEE Transactions Automatic Control, 58(9), 2426-2431, 2013.
- [17] Z. Lu, *Iterative Hard Thresholding Methods for l_0 Regularized Convex Cone Programming*, Mathematical Programming, DOI: 10.1007/s10107-013-0714-4, 2013.
- [18] Z. Lu, L. Xiao, *Randomized Block Coordinate Non-Monotone Gradient Method for a Class of Nonlinear Programming*, Technical Report, [\protect\vrulewidth0pthttp://arxiv.org/abs/1306.5918](http://arxiv.org/abs/1306.5918) , 2013.
- [19] Z. Lu and Y. Zhang, *Sparse Approximation via Penalty Decomposition Methods*, SIAM Journal on Optimization, 23(4), 24482478, 2013.
- [20] A. Monticelli, *Electric power system state estimation*, Proceedings of IEEE, 88(2), 262282, 2000.
- [21] M. Nagahara, D. E. Quevedo, and J. stergaard, *Sparse Packetized Predictive Control for Networked Control over Erasure Channels*, IEEE Transactions on Automatic Control (to appear), 2014.
- [22] M. Nagahara, D. E. Quevedo, and J. Ostergaard, *Packetized predictive control for rate-limited networks via sparse representation*, IEEE Conference on Decision and Control, 1362-1367, 2012.
- [23] I. Necoara, *Random coordinate descent algorithms for multi-agent convex optimization over networks*, IEEE Transactions on Automatic Control, 58(8), 2001-2012, 2013.
- [24] I. Necoara, Y. Nesterov and F. Glineur, *A random coordinate descent method on large optimization problems with linear constraints*, International Conference on Continuous Optimization (ICCOPT), Lisbon, [\protect\vrulewidth0pthttp://acse.pub.ro/person/ion-necoara](http://acse.pub.ro/person/ion-necoara) , 2013.
- [25] I. Necoara and D. N. Clipici, *Distributed coordinate descent methods for composite minimization*, Technical report, University Politehnica Bucharest, [\protect\vrulewidth0pthttp://arxiv-web.arxiv.org/abs/1312.5302](http://arxiv-web.arxiv.org/abs/1312.5302) , 2013.
- [26] I. Necoara and A. Patrascu, *A random coordinate descent algorithm for optimization problems with composite objective function and linear coupled constraints*, Computational Optimization and Applications, 57(2), 307-337, 2014.
- [27] Y. Nesterov, *Efficiency of Coordinate Descent Methods on Huge-Scale Optimization Problems*, SIAM Journal on Optimization 22(2), 341-362, 2012.
- [28] A. Y. Ng, *Feature selection, ℓ_1 vs. ℓ_2 regularization, and rotational invariance*, In Proceedings of the 21st International Conference on Machine learning (ICML), 72-85, 2004.
- [29] A. Patrascu and I. Necoara, *Efficient random coordinate descent algorithms for large-scale structured nonconvex optimization*, Journal of Global Optimization, DOI: 10.1007/s10898-014-0151-9, 2014.
- [30] M. Nikolova, *Description of the minimizers of least squares regularized with 0 norm. Uniqueness of the global minimizer*, SIAM Journal on Imaging Sciences, 6(2), 904-937, 2013.
- [31] P. Richtarik and M. Takac, *Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function*, Mathematical Programming, 144(1-2), 1-38, 2014.